Write the expansion for $\sin n\theta$.

$$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta \ldots\ldots$$

Write the expansion for $\cos n\theta$.

$$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \ldots\ldots$$

Write the expansion for $\tan n\theta$.

$$\tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^3\theta + nc_5 \tan^5\theta \ldots\ldots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta - nc_6 \tan^6\theta + \ldots\ldots}$$

Expansion for $\cos^n\theta$

$$(a+b)^n = a^n + nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 + nc_3 a^{n-3} b^3 + \ldots\ldots$$

Expansion for $\sin^n\theta$

$$(a-b)^n = a^n - nc_1 a^{n-1} b + nc_2 a^{n-2} - nc_3 a^{n-3} b^3 + \ldots\ldots$$

Expansion of $\cos\theta$ in terms of powers of θ.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots$$

Expansion of $\sin\theta$ in terms of powers of θ.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\ldots$$

Expansion of $\tan\theta$ in terms of powers of θ.

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \ldots\ldots$$

Expansion for $\log_e z$

$$\text{Log}_e z = 2n\pi i + \log_e z$$

Expansion for $\log(x+iy)$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1} y/x$$

Find the value of $\text{Log}\, i\,(1+i)$

$$\text{Log}(x+iy) = 2\pi n i + \log(x+iy)$$

$$\text{Log}\, i = 2\pi n i + \log i$$

$$\text{Log}\, i = 2\pi n i + \log(\cos \pi/2 + i \sin \pi/2)$$

$$\text{Log}\, i = 2\pi n i + \log(e^{i\pi/2})$$

$$\text{Log}\, i = 2\pi n i + i\pi/2$$

$$\text{Log}\, i = \pi i/2\,(4n+1)\,.//$$

Find the value of $\text{Log}\, 3i$

$$\text{Log}(x+iy) = 2\pi n i + \log(x+iy)$$

$$\text{Log}\, i = 2\pi n i + \log 3i$$

$$= 2\pi n i + \log 3(\cos \pi/2 + i \sin \pi/2)$$

$$= 2\pi n i + \log 3 + \log e^{i\pi/2}$$

$$= 2\pi n i + \log 3 + i\pi/2$$

$$= \log 3 + i\pi/2\,(4n+1)\,.//$$

prove that $\cos h^2 x - \sin h^2 x = 1$

$$\cos h^2 x - \sin h^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - (e^{2x} + e^{-2x} - 2e^x e^{-x})}{4}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4}$$

$$= 1$$

Prove that $\sec h^2 x + \tan h^2 x = 1$

$$\sec h^2 x + \tan h^2 x = \left(\frac{2}{e^x + e^{-x}}\right)^2 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \frac{4 + (e^x - e^{-x})^2}{e^x + e^{-x} + e^x + e^{-x}}$$

$$= \frac{4 + (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$= \frac{4 + e^{2x} + e^{-2x} - 2e^x e^{-x}}{e^{2x} + e^{-2x} + 2e^x e^{-x}}$$

$$= \frac{e^{2x} + e^{-2x} + 2}{e^{2x} + e^{-2x} + 2}$$

$\sec h^2 x + \tan h^2 x = 1$ .//

$\sin h\,(x \pm y) = \sin h\,x \ \cos h\,y \pm \cos h\,x \ \sin h\,y$

$\sin h(x \pm y) = \frac{1}{i} \sin i\,(x \pm y)$

$= \frac{1}{i} \sin\,(ix \pm iy)$

$= \frac{1}{i} \left[\sin ix \ \cos iy \pm \cos ix \ \sin iy\right]$

$= \frac{1}{i} \left[i \sin h\,x \ \cos h\,y \pm \cos h\,x \ i \sin h\,y\right]$

$= \frac{\cancel{i}}{\cancel{i}} \left[\sin h\,x \ \cos h\,y \pm \cos h\,x \ \sin h\,y\right]$ //.

$1 - \tan h^2 x = \sec h^2 x$

$1 + \tan^2 \theta = \sec^2 \theta$

Put $\theta = ix$

$1 + \tan^2 (ix) = \sec^2 (ix)$

$1 + (i \tan h\,x)^2 = (\sec h\,x)^2$

$1 - \tan h^2 x = \sec h^2 x$ .//

using the following, formulae derive the corresponding hyperbolic function:

i) $$\frac{1+\cos 2x}{2} = \cos^2 x$$

Take $x = ix$

$$\frac{1+\cos 2(ix)}{2} = \cos^2 (ix)$$

$$\frac{1+\cos i(2x)}{2} = \cos^2 (ix)$$

$$\frac{1+\cosh (2x)}{2} = \cosh^2 x \;//$$

ii) $$\frac{1-\cos 2x}{2} = \sin^2 x$$

Replace by $x = ix$

$$\frac{1-\cos 2(ix)}{2} = \sin^2 (ix)$$

$$\frac{1-\cosh 2x}{2} = (i \sinh x)^2 = \frac{1-\cosh 2x}{2} = -\sinh^2 x$$

$$\cancel{+}\frac{(\cosh 2x - 1)}{2} = \cancel{+}\sinh^2 x$$

$$\frac{\cosh 2x - 1}{2} = \sinh^2 x \; ://$$

Separate into real & imaginary of $\cosh(1+i)$

$$\cosh(1+i) = \cos i(1+i)$$

$$= \cos(i-1)$$

$$= \cos i \cos 1 + \sin i \sin 1$$

$$= \cosh \cos 1 + i \sinh \sin 1$$

$$R.P = \cosh \cos 1$$

$$I.P = \sinh \sin 1.$$

S.T $\log(1+i\tan\alpha) = \log(\sec\alpha) + i\alpha.$

$$x = 1, \quad y = \tan\alpha$$

$$\log(1+i\tan\alpha) = \tfrac{1}{2}\log(1^2+\tan^2\alpha) + i\tan^{-1}\left(\frac{\tan\alpha}{1}\right)$$

$$= \tfrac{1}{2}\log \sec^2\alpha + i\alpha$$

$$= \log(\sec^2\alpha)^{1/2} + i\alpha$$

$$= \log(\sec\alpha) + i\alpha$$

Hence proved.//

Find the value of $\log i$

$$\log(x+iy) = 2\pi n i + \log(x+iy)$$

$$\log i = 2\pi n i + \log i$$

$$= 2\pi n i + \log\left(\cos \pi/2 + i \sin \pi/2\right)$$

$$= 2\pi n i + i\pi/2$$

$$= \pi i/2\,(4n+1).$$

Find the value of $\log 3i$

$$\log(x+iy) = 2\pi n i + \log(x+iy)$$

$$\log i = 2\pi n i + \log 3i$$

$$= 2\pi n i + \log 3\left(\cos \pi/2 + i \sin \pi/2\right)$$

$$= 2\pi n i + \log 3\left(e^{i\pi/2}\right)$$

$$= 2\pi n i + \log 3 + \log e^{i\pi/2}$$

$$= 2\pi n i + \log 3 + i\pi/2$$

$$\log 3 + i\pi/2\,(4n+1).$$

Separate the following real & imaginary part.

$\sin(x+iy)$.

$\sin x \cos iy + \cos x \sin iy$.

$\sin x \cosh y + \cos x \, i \sinh y$.

Real part $= \sin x \cosh y$.

imaginary part $= \cos x \sinh y$.

Separate the following real & imaginary part

$\cos(x+iy)$.

$\cos x \cos iy - \sin x \sin iy$

$\cos x \cosh y - \sin x \, i \sinh y$.

Real part $= \cos x \cosh y$

imaginary part. $= -\sin x \sinh y$.

prove that :- $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$.

~~$\sinh^2 B$~~

given $\sin(A+iB) = x+iy$.

$\sin A \cos iB + \cos A \sin iB = x+iy$.

$\sin A \cosh B + \cos A \, i \sinh B = x+iy$.

$x = \sin A \cosh B$.

$y = \cos A \sinh B$.

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1.$$

$$\frac{\sin^2 A \cancel{\cosh^2 B}}{\cancel{\cosh^2 B}} + \frac{\cos^2 A \cancel{\sinh^2 B}}{\cancel{\sinh^2 B}}.$$

$\sin^2 A + \cos^2 A$

$= 1.$

Hence proved.

Find the value of $\operatorname{Log}\sqrt{i}$.

$$\begin{aligned}
\operatorname{Log}\sqrt{i} &= 2\pi n i + \log\sqrt{i}. \\
&= 2\pi n i + \log\sqrt{\cos\pi/2 + i\sin\pi/2} \\
&= 2\pi n i + \log\sqrt{e^{i\pi/2}}. \\
&= 2\pi n i + \log\left(e^{i\pi/2}\right)^{1/2}. \\
&= 2\pi n i + \log e^{i\pi/4} \\
&= 2\pi n i + \frac{i\pi}{4} \\
&= \pi i\left(2n + \tfrac{1}{4}\right) \\
&= \frac{\pi i}{4}(8n+1)
\end{aligned}$$

Solve : $6\cos\theta - 8\sin\theta = 7$.

Solution :-

$a = 6$, $b = 8$.

$\sqrt{a^2+b^2} = \sqrt{36+64} = \sqrt{100} = 10$.

Divide the given equation by 10.

$0.6\cos\theta - 0.8\sin\theta = 0.7$.

$\cos 53^\circ 8' \cos\theta - \sin 53^\circ 8' \sin\theta = \cos 45^\circ 37'$.

$\cos(\theta + 53^\circ 8') = \cos 45^\circ 37'$.

Therefore the general solution is.

$\theta + 53^\circ 8' = n.360^\circ \pm 45^\circ 37'$.

$\theta = n.360^\circ - 98^\circ 45'$ (or) $\theta = n.360^\circ - 7^\circ 31'$.

Prove that $\tan^{-1}\frac{x-y}{1+xy} + \tan^{-1}\frac{y-z}{1+yz} + \tan^{-1}\frac{z-x}{1+zx} = 0$.

Solution :-

$\tan^{-1}\frac{x-y}{1+xy} = \tan^{-1}x - \tan^{-1}y$

$\tan^{-1}\frac{y-z}{1+yz} = \tan^{-1}y - \tan^{-1}z$.

$\tan^{-1}\frac{z-x}{1+zx} = \tan^{-1}z - \tan^{-1}x$.

$= \cancel{\tan^{-1}x} - \cancel{\tan^{-1}y} + \cancel{\tan^{-1}y} - \cancel{\tan^{-1}z} + \cancel{\tan^{-1}z} - \cancel{\tan^{-1}x}$

$= 0$.

Hence proved.

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$. Show that $x+y+z = xyz$.

Solution:-

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi.$$

$$\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx} = \pi.$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\pi.$$

$$\boxed{\tan\pi = 0}$$

$$x+y+z-xyz = 0.$$

$$x+y+z = xyz.$$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ Show that $xy+yz+zx = 1$.

Solution:-

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}.$$

$$\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx} = \frac{\pi}{2}$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\frac{\pi}{2} = \infty$$

$$1-xy-yz-zx = 0.$$

(or) $xy+yz+zx = 1$

If $\tan\frac{x}{2} = \tanh\frac{x}{2}$ show that $\cos x \cosh x = 1$.

Solution:-

$$\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}.$$

$$\cos x = \frac{1-\tanh^2 x/2}{1+\tanh^2 x/2}.$$

$$\cos x. = \frac{\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}}{\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}}.$$

$$\cos x = \frac{1}{\cosh x}.$$

$$\cos x \cosh x = 1.$$

Evaluate. $\lim_{\theta\to 0}\left[\frac{\theta-\sin\theta}{\theta^3}\right]$

$$\lim_{\theta\to 0}\left[\frac{\theta-\sin\theta}{\theta^3}\right] = \lim_{\theta\to 0}\left[\frac{\theta-\left[\theta-\theta^3/3!+\cdots\right]}{\theta^3}\right]$$

$$\lim_{\theta\to 0}\left[\frac{\cancel{\theta}-\cancel{\theta}+\theta^3/6+\cdots}{\theta^3}\right]$$

$$\lim_{\theta\to 0}\left[\frac{\cancel{\theta^3}}{6\cancel{\theta^3}}\right]$$

$$= \frac{1}{6}.$$

$\coth^2 x - \operatorname{cosech}^2 x = 1$. prove that.

$$\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - \left(\frac{2}{e^x - e^{-x}}\right)^2 = 1.$$

$$= \frac{e^{2x} + e^{-2x} + 2e^x e^{-x} - 4}{e^{2x} + e^{-2x} - 2e^x e^{-x}}.$$

$$= \frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} - 2}.$$

$$= 1$$

Hence proved.

$\sinh 2x = 2\sinh x \cosh x$.

$$\sinh 2x = \frac{1}{i} \sin i2x.$$

$$= \frac{1}{i} \sin 2(ix)$$

$$= \frac{1}{i} 2\sin ix \cos ix.$$

$$= \frac{1}{\cancel{i}} 2\cancel{i} \sinh x \cosh x.$$

$\sinh 2x. = 2\sinh x \cosh x.$

hence proved.

If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ find θ approximately.

$$\frac{\tan\theta}{\theta} = \frac{2524}{2523}.$$

$$\frac{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots}{\theta} = 1 + \frac{1}{2523}.$$

$$1 + \frac{\theta^2}{3} + \cdots = 1 + \frac{1}{2523}$$

$$\frac{\theta^2}{3} = \frac{1}{2523}.$$

$$\theta^2 = \frac{3}{2523}.$$

$$\theta^2 = \frac{1}{841}$$

$$\theta' = \frac{1}{29}.$$

$$\theta = 1^\circ 58'.$$

Prove the $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$.

Chage x into ix.

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

$$\cos 2ix = \frac{1-(\tan ix)^2}{1+(\tan ix)^2}$$

$$\cosh 2x = \frac{1-i^2\tanh^2 x}{1+i^2\tanh^2 x} = \frac{1+\tanh^2 x}{1-\tanh^2 x}.$$

Hence proved.

Prove the $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$.

Change x into ix.

$$\tan(3ix) = \frac{3\tan ix - (\tan ix)^3}{1-3(\tan ix)^2}.$$

$$i\tanh 3x = \frac{3i\tanh x - i^3\tanh^3 x}{1-3i^2\tanh^2 x}.$$

$$i\tanh 3x = \frac{i[3\tanh x + \tanh^3 x]}{1+3\tanh^2 x}.$$

Cancelling i on both the sides.

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}.$$

Expand $\tan 7\theta$ in terms of $\tan\theta$.

Solution :-

$$\tan 7\theta = \frac{7C_1 \tan\theta - 7C_3 \tan^3\theta + 7C_5 \tan^5\theta - \tan^7\theta}{1 - 7C_2 \tan^2\theta + 7C_4 \tan^4\theta - 7C_6 \tan^6\theta}$$

$$= \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

Evaluate $\underset{x \to 0}{Lt} \frac{\tan 2x - 2\tan x}{x^3}$

Solution :-

$$\underset{x \to 0}{Lt} \frac{\tan 2x - 2\tan x}{x^3}.$$

$$= \underset{x \to 0}{Lt} \frac{2x + \frac{(2x)^3}{3} + \frac{2}{15}(2x)^5 + \cdots - 2\left(x + \frac{x^3}{3} + \cdots\right)}{x^3}.$$

$$= \underset{x \to 0}{Lt} \frac{\left(\frac{8x^3}{3} - 2\,\frac{x^3}{3}\right) + \left(\frac{64}{15}x^5 - \frac{4}{15}x^5\right) + \cdots \infty}{x^3}.$$

$$= \underset{x \to 0}{Lt} \left[2 + 4x^2 + \cdots \infty\right] = 2.$$

If $\frac{\sin x}{x} = \frac{863}{864}$ find an approximate value of x.

Solution :-

$$\frac{\sin x}{x} = \frac{863}{864}.$$

$$\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = \frac{863}{864}.$$

$$\frac{\cancel{x}\left(1 - \frac{x^2}{3!} + \cdots\right)}{\cancel{x}} = \frac{863}{864}.$$

$$\cancel{1} - \frac{x^2}{6} = \cancel{1} - \frac{1}{864}$$

$$\frac{x^2}{6} = \frac{1}{864}.$$

$$x^2 = \frac{6}{864}.$$

$$x^2 = \frac{1}{144}$$

$$x = \frac{1}{12} = \frac{1}{12} \times 57^\circ 17' 44''$$

$$x = 4^\circ 46' 29''$$

If $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, show that $x = \frac{a-b}{1+ab}$.

Solution:-

$$\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$$

$$2\tan^{-1}a - 2\tan^{-1}b = 2\tan^{-1}x.$$

$$\tan^{-1}a - \tan^{-1}b = \tan^{-1}x$$

$$\tan^{-1}\frac{a-b}{1+ab} = \tan^{-1}x.$$

$$x = \frac{a-b}{1+ab}.$$

Solve : $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

Solution:-

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}1 - \tan^{-1}\left(\frac{x+1}{x+2}\right).$$

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}$$

$$= \tan^{-1}\left(\frac{x+2-x-1}{x+2+x+1}\right).$$

$$= \tan^{-1}\left(\frac{1}{2x+3}\right).$$

$$\frac{x-1}{x-2} = \frac{1}{2x+3}$$

$$(x-1)(2x+3) = x-2$$

$$2x^2 = 1 \quad (\text{or}) \quad x = \pm\frac{1}{\sqrt{2}}.$$

Solve :- $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$

Solution:-

$$3(2\tan^{-1}x) - 4(2\tan^{-1}x) + 2(2\tan^{-1}x) = \frac{\pi}{3}.$$

$$2\tan^{-1}x = \frac{\pi}{3}.$$

$$\tan^{-1}x = \frac{\pi}{6}.$$

$$x = \tan\frac{\pi}{6}.$$

$$x = \frac{1}{\sqrt{3}}.$$

Solve : $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$

Solution:-

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}3x - \tan^{-1}(x)$$

$$\tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{3x-x}{1+3x^2}.$$

$$\frac{2x}{2-x^2} = \frac{2x}{1+3x^2}.$$

$$\therefore\ x = 0 \text{ (or) } 4x^2 = 1$$

$$\therefore\ x = 0, \pm\frac{1}{2}.$$

Solve :- $\sin x + \cos x = 1$.

Solution :-

$\sin x + \cos x = 1$.

Dividing by $\sqrt{2}$,

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = \frac{1}{\sqrt{2}}$$

$$\sin \frac{\pi}{4} \sin x + \cos \frac{\pi}{4} \cos x = \cos \pi/4$$

$$\cos\left(x - \frac{\pi}{4}\right) = \cos \pi/4$$

$$x - \frac{\pi}{4} = 2n\pi \pm \pi/4$$

$$x = 2n\pi \text{ (or) } 2n\pi + \pi/2 \quad , \text{ where } n = 0, \pm 1, \pm 2, \ldots$$

Solve :- $\tan x - \sqrt{2} \sec x = \sqrt{3}$.

Solution :-

$$\frac{\sin x}{\cos x} - \sqrt{2}\left(\frac{1}{\cos x}\right) = \sqrt{3}.$$

$$\frac{\sin x - \sqrt{2}}{\cos x} = \sqrt{3}$$

$$\sin x - \sqrt{2} = \sqrt{3} \cos x.$$

$$\sin x - \sqrt{3} \cos x = \sqrt{2}.$$

Dividing by 2,

$$\frac{1}{2} \sin x - \frac{\sqrt{3}}{2} \cos x = \frac{1}{2} \sqrt{2}.$$

$$\sin \frac{\pi}{6} \sin x - \cos \frac{\pi}{6} \cos x = -\frac{1}{\sqrt{2}}.$$

$$\cos\left(x + \frac{\pi}{6}\right) = \cos \frac{3\pi}{4}$$

$$x + \frac{\pi}{6} = 2n\pi \pm \frac{3\pi}{4}$$

$$x = 2n\pi + 7\pi/12 \, , \; 2n\pi - 11\pi/12 \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

Solve :- $\sec^2\theta - (\sqrt{3}+1)\tan\theta + \sqrt{3} - 1 = 0$.

Solution :-

$$1 + \tan^2\theta - (\sqrt{3}+1)\tan\theta + \sqrt{3} - 1 = 0$$

$$\tan^2\theta - (\sqrt{3}+1)\tan\theta + \sqrt{3} = 0$$

$\sqrt{3}\tan^2\theta$

$-1\tan\theta \,/\, \tan^2\theta$

$-\sqrt{3}\tan\theta \,/\, \tan^2\theta$.

$-(\sqrt{3}+1)\tan\theta$

$$(\tan\theta - 1)(\tan\theta - \sqrt{3}) = 0$$

$\tan\theta - 1 = 0$	$\tan\theta - \sqrt{3} = 0$
$\tan\theta = 1$	$\tan\theta = \sqrt{3}$.

$\tan\theta = 1$, gives the general solution

$$\theta = n\pi + \frac{\pi}{4}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$\tan\theta = \sqrt{3}$ gives the general solution

$$\theta = n\pi + \frac{\pi}{3}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Solve : $2\sin^2 x - 3\cos x - 3 = 0$

Solution :-

$$2(1-\cos^2 x) - 3\cos x - 3 = 0$$

$$2 - 2\cos^2 x - 3\cos x - 3 = 0.$$

$$-2\cos^2 x - 3\cos x - 1 = 0$$

$$2\cos^2 x + 3\cos x + 1 = 0.$$

$$2\cos^2 x \begin{cases} 1\cos x \,/\, 2\cos^2 x \\ 2\cos x \,/\, 2\cos^2 x. \end{cases}$$

$$\underline{\overline{3\cos x}}.$$

$$(2\cos x + 1)(\cos x + 1) = 0.$$

$2\cos x + 1 = 0.$	$\cos x + 1 = 0$
$2\cos x = -1$	$\cos x = -1$
$\cos x = \frac{-1}{2}.$	

Therefore $\cos x = -\frac{1}{2}$ (or) -1.

$\cos x = -\frac{1}{2}$ the general solution is.

$$\theta = 2n\pi \pm \frac{2\pi}{3}, \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$$

$\cos x = -1$ the general solution is.

$$\theta = 2n\pi \pm \pi.$$

$$n = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$$

Solve : $2\sin x + \operatorname{cosec} x = 3$.

Solution :-

$$2\sin x + \frac{1}{\sin x} = 3.$$

$$2\sin x - 3 + \frac{1}{\sin x} = 0$$

$$\frac{2\sin^2 x - 3\sin x + 1}{\sin x} = 0.$$

$$2\sin^2 x - 3\sin x + 1 = 0$$

$2\sin^2 x$ < $-1\sin x \,/\, 2\sin^2 x$; $-2\sin x \,/\, 2\sin^2 x$

$-3\sin x$

$$(2\sin x - 1)(\sin x - 1) = 0$$

Therefore $\sin x = \frac{1}{2}$ (or) 1

$\sin x = 1/2$ the general solution is.

$$x = n\pi + (-1)^n \pi/6 \;,\; n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$\sin x = 1$ the general solution is

$$x = n\pi + (-1)^n \pi/2 \;,\; n = 0, \pm 1, \pm 2, \ldots$$

Solve : $\sqrt{2}\sec\theta + \tan\theta = 1$.

Solution:-

$$\frac{\sqrt{2}}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = 1.$$

$$\frac{\sqrt{2} + \sin\theta}{\cos\theta} = 1.$$

$$\sqrt{2} + \sin\theta = \cos\theta$$

$$\sqrt{2} = \cos\theta - \sin\theta$$

$$\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = 1$$

$$\cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\theta = 2n\pi + \frac{\pi}{2} \text{ (or) } 2n\pi.$$

Solve the equation $\tan 3\theta - 4\tan\theta = 0$.

Solution:-

$\boxed{\tan\theta = t}$. · Applying the $\tan n\theta$ formula

$$\frac{3t - t^3}{1 - 3t^2} - 4t = 0.$$

$$\frac{3t - t^3 - 4t(1 - 3t^2)}{1 - 3t^2} = 0.$$

$$3t - t^3 - 4t(1 - 3t^2) = 0.$$

$$t(3 - t^2 - 4(1 - 3t^2)) = 0. \Rightarrow 3 - t^2 - 4 + 12t^2 = 0 \Rightarrow 11t^2 = 1.$$

$\therefore\ t=0$ (or) $11t^2=1$.

$\tan\theta=0$ (or) $\tan\theta=\pm\dfrac{1}{\sqrt{11}}$ (multiple and diving $\sqrt{11}$).

$$\tan\theta=\frac{\sqrt{11}}{11}=\frac{3.317}{11}.$$

$$\tan\theta=0.3015.$$

$\theta=n\pi(1)$ (or) $\alpha=\pm 16^\circ 47'$.

$\therefore\ \theta=\pm 16^\circ 47'+n180$.

where n is a $\pm$ integer or 0.

Solve the equation $2\sin^2 x+4\sin x\cos x=3$.

Solution :-

Dividing by $\cos^2 x$, we have.

$$2\frac{\sin^2 x}{\cos^2 x}+\frac{4\sin x\cos x}{\cos^2 x}=\frac{3}{\cos^2 x}$$

$$2\tan^2 x+4\tan x=3\sec^2 x.$$

$$2\tan^2 x+4\tan x=3(1+\tan^2 x)$$

$$2\tan^2 x+4\tan x=3+3\tan^2 x.$$

$$3+3\tan^2 x-2\tan^2 x-4\tan x.$$

$$\tan^2 x-4\tan x+3=0.$$ $3 \begin{matrix} 1 \\ 3 \end{matrix}$ $\frac{}{4}$

$$(\tan x-1)(\tan x-3)=0.$$

$\tan x=1$ (or) 3.

$\alpha=\pi/4$, $\therefore\ x=\pi/4+n\pi$.

$x=71^\circ 30'+n180^\circ$. where n is a $\pm$ integer or 0.

Expand $\cos 8\theta$ in terms of power of cosine.

FORMULA:

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta - \cdots$$

$n = 8$

$$\cos 8\theta = \cos^8\theta - 8C_2 \cos^6\theta \sin^2\theta + 8C_4 \cos^4\theta \sin^4\theta - 8C_6 \cos^2\theta \sin^6\theta + 8C_8 \cos^0\theta \sin^8\theta$$

$$= \cos^8\theta - 28\cos^6\theta \sin^2\theta + 70\cos^4\theta \sin^4\theta - 28\cos^2\theta \sin^6\theta + \sin^8\theta$$

$$= \cos^8\theta - 28\cos^6\theta (1-\cos^2\theta) + 70\cos^4\theta (1 - 2\cos^2\theta + \cos^4\theta) - 28\cos^2\theta (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) + \sin^8\theta.$$

$$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta - 140\cos^6\theta + 70\cos^8\theta - 28\cos^2\theta + 84\cos^4\theta - 84\cos^6\theta + 28\cos^8\theta + \sin^8\theta$$

$$= \cos^8\theta - 28\cos^6\theta + 28\cos^8\theta + 70\cos^4\theta - 140\cos^6\theta + 70\cos^8\theta - 28\cos^2\theta + 84\cos^4\theta - 84\cos^6\theta + 28\cos^8\theta + 1 - 4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta$$

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 16\cos^4\theta - 32\cos^2\theta + 1$$

EXPAND $\cos 9\theta$ in terms of power of cosine

FORMULA

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta - nC_6 \cos^{n-6}\theta \sin^6\theta + \ldots$$

$n = 9$

$$\cos 9\theta = \cos^9\theta - 9C_2 \cos^7\theta \sin^2\theta + 9C_4 \cos^5\theta \sin^4\theta - 9C_6 \cos^3\theta \sin^6\theta + 9C_8 \cos\theta \sin^8\theta$$

$$= \cos^9\theta - 36 \cos^7\theta \sin^2\theta + 126 \cos^5\theta \sin^4\theta - 84 \cos^3\theta \sin^6\theta + 9 \cos\theta \sin^8\theta$$

$$= \cos^9\theta - 36 \cos^7\theta (1-\cos^2\theta) + 126 \cos^5\theta (1-2\cos^2\theta + \cos^4\theta) - 84 \cos^3\theta (1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) + 9\cos\theta (1-4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta)$$

$$= \cos^9\theta - 36\cos^7\theta + 36\cos^9\theta + 126 \cos^5\theta - 252 \cos^7\theta + 126 \cos^9\theta - 84\cos^3\theta + 252\cos^5\theta - 252 \cos^7\theta +$$

$84\cos^9\theta + 9\cos\theta - 36\cos^3\theta + 54\cos^5\theta - 36$

$\cos^7\theta + 9\cos^9\theta$

$\Rightarrow$

$$\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta$$
$$-120\cos^3\theta + 9\cos\theta.$$

EXPAND $\cos 5\theta$ in terms of power of casine

FORMULA

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4$$
$$\cos^{n-4}\theta \sin^4\theta - \cdots$$

$n = 5$

$$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta +$$
$$5C_4 \cos\theta \sin^4\theta.$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta$$
$$\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta$$
$$(1 - 2\cos^2\theta + \cos^4\theta)$$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta$

$+ 5\cos^5\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

EXPAND $\sin 7\theta$ in terms of powers of sines.

FORMULA.

$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta$
$+ nC_5 \cos^{n-5}\theta \sin^5\theta - nC_7 \cos^{n-7}\theta \sin^7\theta$
$+ \cdots$

$n = 7$

$\sin 7\theta = 7C_1 \cos^{7-1}\theta \sin\theta - 7C_3 \cos^4\theta \sin^3\theta$
$+ 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \cos^0\theta \sin^7\theta$

$= 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21$
$\cos^2\theta \sin^5\theta - (1)\sin^7\theta$

$= 7\sin\theta(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)$
$- 35\sin^3\theta(1 - 2\sin^2\theta + \sin^4\theta) +$
$21\sin^5\theta(1 - \sin^2\theta) - \sin^7\theta$

$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta -$
$35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta$
$- 21\sin^7\theta - \sin^7\theta$

$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta$
$+ 7\sin\theta.$

SHOW THAT $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta$
$- 64\sin^6\theta$

FORMULA

$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta$
$+ nC_5 \cos^{n-5}\theta \sin^5\theta - nC_7 \cos^{n-7}\theta \sin^7\theta$
$+ \cdots$

$n = 7$

$\sin 7\theta = 7C_1 \cos^6\theta \sin\theta - 7C_3 \cos^4\theta \sin^3\theta$
$+ 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \cos^0\theta \sin^7\theta$

$= 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta$
$\sin^5\theta - (1)\sin^7\theta$
$= 7(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin\theta -$
$35\sin^3\theta(1 - 2\sin^2\theta + \sin^4\theta) +$
$21\sin^5\theta(1 - \sin^2\theta) - \sin^7\theta$

$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35$
$\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta$
$- \sin^7\theta$

$\sin 7\theta = -64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta$

Divide $\sin\theta$ on Both side

$$\frac{\sin 7\theta}{\sin\theta} = \frac{-64\sin^7\theta + 112\sin^5\theta - 56\sin^3\theta + 7\sin\theta}{\sin\theta}$$

$$\frac{\sin 7\theta}{\sin\theta} = -64\sin^6\theta + 112\sin^4\theta - 56\sin^2\theta + 7$$

HENCE PROVED.

SHOW THAT $\dfrac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$

FORMULA

$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta$

$n = 7$

$\cos 7\theta = \cos^7\theta - 7C_2\cos^5\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta - 7C_6\cos\theta\sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta \sin^2\theta + 35\cos^3\theta \sin^4\theta - 7 \cos\theta \sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta (1-\cos^2\theta) + 35\cos^3\theta (1-2\cos^2\theta + \cos^4\theta) - 7\cos\theta (1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$

$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Divide $\cos\theta$ on Both Side

$$\frac{\cos 7\theta}{\cos\theta} = \frac{64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta}{\cos\theta}$$

$$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

HENCE PROVED.

$$\frac{\cos 5\theta}{\cos\theta}$$

FORMULA

$\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \cdots$

$n = 5$

$$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta + 5C_4 \cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta (1-\cos^2\theta) + 5\cos\theta (1 - 2\cos^2\theta + \cos^4\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

Divide $\cos\theta$ on Both Side

$$\frac{\cos 5\theta}{\cos\theta} = \frac{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}{\cos\theta}$$

$$\frac{\cos 5\theta}{\cos\theta} = 16\cos^4\theta - 20\cos^2\theta + 5$$

HENCE PROVED.

$\frac{\sin 5\theta}{\sin\theta}$.

FORMULA:

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + nC_5 \cos^{n-5}\theta \sin^5\theta - \dots$$

$n = 5$

$\sin 5\theta = \; 5C_1 \cos^4\theta \sin\theta - 5C_3 \cos^2\theta \sin^3\theta + 5C_5$
$\cos^0\theta \sin^5\theta$

$= 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + 1 \sin^5\theta$

$= 5\sin\theta (1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta$
$(1 - \sin^2\theta) + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta$
$+ \sin^5\theta$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Divide $\sin\theta$ on Both Side.

$$\frac{\sin 5\theta}{\sin\theta} = \frac{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}{\sin\theta}$$

$$\frac{\sin 5\theta}{\sin\theta} = 16\sin^4\theta - 20\sin^2\theta + 5$$

Express $\cos^5\theta$ terms of Cosine of multiple angles?

$x + 1/x = 2\cos\theta$

$\left[x + \frac{1}{x}\right]^5 = [2\cos\theta]^5$

FORMULA

$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2$
$+ nC_3 a^{n-3} b^3 + \dots$

$$2^5 \cos^5\theta = x^5 + 5C_1\, x^{5-1}\left[\frac{1}{x}\right] + 5C_2\, x^{5-2}\left[\frac{1}{x}\right]^2 + 5C_3\, x^{5-3}\left[\frac{1}{x}\right]^3 + 5C_4\, x^{5-4}\left[\frac{1}{x}\right]^4 + 5C_5\, x^{5-5}\left[\frac{1}{x}\right]^5$$

$$= x^5 + 5x^4 \times \frac{1}{x} + 10\, x^3 \times \frac{1}{x^2} + 10x^2 \times \frac{1}{x^3} + 5x \times \frac{1}{x^4} + \frac{1}{x^5}$$

$$= x^5 + 5x^3 + 10x + 10 \times \frac{1}{x} + 5 \times \frac{1}{x^3} + \frac{1}{x^5}$$

$$= \left[x^5 + \frac{1}{x^5}\right] + 5\left[x^3 + \frac{1}{x^3}\right] + 10\left[x + \frac{1}{x}\right]$$

$$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$$

$$2^5 \cos^5\theta = 2\left[\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right]$$

$$\cos^5\theta = \frac{1}{2^4}\left[\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right]$$

Express $\cos^6\theta$ terms of cosine of multiple angles

$$x + \frac{1}{x} = 2\cos\theta$$

$$\left[x + \frac{1}{x}\right]^6 = \left[2\cos\theta\right]^6$$

FORMULA

$$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + nC_3 a^{n-3} b^3$$

$n = 6 \quad a = x \quad b = 1/x$

$$2^6 \cos^6\theta = x^6 + 6C_1 x^{6-1}\left[\frac{1}{x}\right] + 6C_2 x^{6-2}\left[\frac{1}{x^2}\right] + 6C_3 x^{6-3}\left[\frac{1}{x^3}\right] + 6C_4 x^{6-4}\left[\frac{1}{x^4}\right] + 6C_5 x^{6-5}\left[\frac{1}{x^5}\right] + 6C_6 x^{6-6}\left[\frac{1}{x^6}\right]$$

$$= x^6 + 6x^5 \times \frac{1}{x} + 15x^4 \times \frac{1}{x^2} + 20x^3 \times \frac{1}{x^3} + 15x^2 \times \frac{1}{x^4} + 6x \times \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15 \times \frac{1}{x^2} + 6 \times \frac{1}{x^4} + \frac{1}{x^6}$$

$$= \left[x^6 + \frac{1}{x^6}\right] + 6\left[x^4 + 1/x^4\right] + 15\left[x^2 + \frac{1}{x^2}\right] + 20$$

$$2^6 \cos^6\theta = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$2^6 \cos^6\theta = 2\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$$

$$\cos^6\theta = \frac{1}{2^5}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right]$$

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

Soln :-

$$\tan \theta = \theta + \frac{\theta^3}{3} + 2\frac{\theta^5}{15} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + 2\frac{x^5}{15} + \cdots$$

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \left[\frac{x + \frac{x^3}{3} + 2\frac{x^5}{15} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right)}{x^3} \right]$$

$$= \lim_{x \to 0} \left[\frac{x + \frac{x^3}{3} + 2\frac{x^5}{15} - x + \frac{x^3}{3!} - \frac{x^5}{5!}}{x^3} \right]$$

$$= \lim_{x \to 0} \cancel{x^3} \left[\frac{\cancel{x} + 1/3 + 2\frac{x^2}{15} - \cancel{x} + \frac{1}{3!} - \frac{x^2}{5!}}{\cancel{x^3}} \right]$$

$$= \lim_{x \to 0} \left[\frac{1}{3} + 2\frac{x^2}{15} + \frac{1}{6} - \frac{x^2}{120} \right]$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{6+3}{3} = \frac{9}{3} = 1/2 \, //$$

Proved

~x~

$$\lim_{\theta \to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

Soln :-

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \tan x = x + \frac{x^3}{3} + 2\frac{x^5}{15} + \cdots$$

$$\lim_{\theta \to 0}\left[\frac{\tan x - \sin x}{\sin^3 x}\right] = \lim_{\theta \to 0}\left[\frac{x + \frac{x^3}{3} + 2\frac{x^5}{15} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3}\right]$$

$$= \lim_{\theta \to 0}\left[\frac{\cancel{x} + \frac{x^3}{3} + 2\frac{x^5}{15} - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!}}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)^3}\right]$$

$$= \lim_{\theta \to 0}\left[\frac{x^3\left(\frac{1}{3} + 2\frac{x^2}{15} + \frac{1}{3!} - \frac{x^2}{5!}\right)}{x^3\left(1 - \left(\frac{x^6}{(6)^3}\right) + \frac{x^{12}}{(120)^3}\right)}\right]$$

$$= \frac{1}{3} + \frac{1}{6}$$

$$= \frac{9}{18}$$

$= 1/2$ Proved //

~ x ~

$$\lim_{x \to \pi/2} (\sec x - \tan x)$$

Soln :-

Take $x \to \pi/2 + \theta$ Here $\theta = 0$

i.e $x \to \pi/2$ & $\theta \to 0$

$$\lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2 + 0} \left[\sec(\pi/2 + 0) - \tan(\pi/2 + 0) \right]$$

$$= \lim_{x \to \pi/2 + 0} \left[\left(\frac{1}{\cos(\pi/2 + 0)} \right) - (-\cot \theta) \right]$$

$$= \lim_{x \to \pi/2 + 0} \left[\frac{-1}{\sin \theta} + \cot \theta \right]$$

$$= \lim_{x \to \pi/2 + 0} \left[\frac{-1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{-1 + \cos \theta}{\sin \theta} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{-1 + \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \right)}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{-1 + 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\cancel{\theta^2}\left[\frac{-1}{2!} + \frac{\theta^2}{4!}\right]}{\cancel{\theta}\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}\right]} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\theta\left(\frac{-1}{2!} + \frac{\theta^2}{4!}\right)}{\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}\right)} \right]$$

$$= 0$$

~ x ~

If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$ Prove that the angle θ is $1^\circ 58'$ nearly.

Soln :-

W.K.T

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$\sin\theta = \theta\left[1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right]$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!}$$ [nighting the higer Powers of θ]

$$\frac{5045}{5046} = 1 - \frac{\theta^2}{3!}$$

$$\cancel{1} - \frac{\theta^2}{6} = \cancel{1} - \frac{5045}{5046}$$

$$\frac{\theta^2}{6} = \frac{1}{5046}$$

$$\theta^2 = \frac{6}{5046}$$

$$\theta^2 = \frac{1}{841}$$

$$\theta = \sqrt{1/841}$$

$$\theta = \frac{1}{29} \text{ radians}$$

$$\theta = \frac{1}{29} \times \frac{180}{\pi} \text{ degree}$$ 1 radian $= \frac{180}{\pi}$ degree

$$= \frac{1}{29} \times \frac{180 \times 7}{22}$$

$$= \frac{1}{29} \times \frac{1260}{22}$$

$$= \frac{1}{29} \times \frac{630}{11}$$

$$= 1.9742$$

$$= 1^\circ.58' \text{ nearly}$$ // Hence Proved

~ x ~

If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$ Prove that the angle θ is

P.T $\theta = 3^\circ$

Soln :-

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right)$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots$$

$$\frac{2165}{2166} = 1 - \frac{\theta^2}{3!}$$

$$\cancel{1} - \frac{\theta^2}{6} = \cancel{1} - \frac{1}{2166}$$

$$\frac{\theta^2}{6} = \frac{1}{2166}$$

$$\theta^2 = \frac{6}{2166}$$

$$\theta^2 = \frac{1}{361} \Rightarrow \theta = \sqrt{1/361} \Rightarrow \theta = 1/19 \text{ radians}$$

$$\theta = \frac{1}{19} \times \frac{180}{\pi}$$

$$= \frac{1}{19} \times \frac{180 \times 7}{22}$$

$$= \frac{1}{19} \times \frac{1260}{22}$$

$$= \frac{1}{19} \times \frac{630}{11}$$

$$= 3.0$$

$\theta = 3$ Hence proved.

~ x ~

If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ Prove that the angle θ is P.T $\theta = 1^\circ 58'$

Soln :-

$$\tan\theta = \theta + \frac{\theta^3}{3} + 2\frac{\theta^5}{15} + \cdots$$

$$\tan\theta = \theta\left(1 + \frac{\theta^2}{3} + 2\frac{\theta^4}{15} + \cdots\right)$$

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + 2\frac{\theta^4}{15}$$

$$\frac{2524}{2523} = 1 + \frac{\theta^2}{3}$$

$$\cancel{1} + \frac{\theta^2}{3} = \cancel{1} + \frac{1}{2523}$$

$$\frac{\theta^2}{3} = \frac{1}{2523}$$

$$\theta^2 = \frac{3}{2523}$$

$$\theta^2 = \frac{1}{841}$$

$$\theta = \sqrt{1/841}$$

$$\theta = 1/29 \text{ radions}$$

$$\theta = \frac{1}{29} \times \frac{180 \times 7}{22}$$

$$= \frac{1}{29} \times \frac{2160}{22}$$

$$= \frac{1}{29} \times \frac{630}{11}$$

$$\begin{array}{r} .9742 \times 60 \\ \hline 58.452 \end{array}$$

$$= 1.9742$$

$$= 1^\circ .58' \quad \text{Hence proved.}$$

~ × ~

If $\cos\theta = \frac{1681}{1682}$ prove that the angle θ is

P.T $\theta = 1^\circ.58'$ nearly

Soln :-

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

$$\frac{1681}{1682} = 1 - \frac{\theta^2}{2!}$$

$$\cancel{1} - \frac{\theta^2}{2!} = \cancel{1} - \frac{1}{1682}$$

$$\frac{\theta^2}{2} = \frac{1}{1682}$$

$$\theta^2 = \frac{2}{1682}$$

$$\theta^2 = 1/841$$

$$\theta = \sqrt{1/841} = 1/29 \text{ radians}$$

$$\theta = \frac{1}{29} \times \frac{180}{\pi}$$

$$= \frac{1}{29} \times \frac{180 \times 7}{22}$$

$$= \frac{1}{29} \times \frac{1260}{22}$$

$$= \frac{1}{29} \times \frac{360}{11}$$

$$= 1.9742$$

$$= 1^{\circ}.58' \text{ nearly}$$

Hence proved.

~ × ~

Solve approximately $\sin\left(\frac{\pi}{6}+\theta\right) = 0.51$

Soln :-

$$\sin\left(\frac{\pi}{6}+\theta\right) = 0.51$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B = 0.51$$

$$\sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = 0.51$$

$$\frac{1}{2}\left[1-\frac{\theta^2}{2!}+\cdots\right] + \frac{\sqrt{3}}{2}\left[\theta-\frac{\theta^3}{3!}+\cdots\right] = 0.51$$

$$\frac{1}{2} - \frac{\theta^2}{2.2} + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51$$

$$\frac{1+\sqrt{3}\theta}{2} = 0.51$$

$$1+\sqrt{3}\theta = 2\times 0.51$$

$$\sqrt{3}\theta = 2\times 0.51 - 1$$

$$\theta = \frac{2\times(0.51-1)}{\sqrt{3}}$$

$$= 0.58 \text{ radion}$$

$$= 0.58\times\frac{180}{\pi} \text{ degree}$$

$= 0$ Hence proved.

Solve that if $\cos\left(\frac{\pi}{3}+\theta\right)=0.49$, θ is $40'$ nearly.

Soln :-

$$\cos\left(\frac{\pi}{3}+\theta\right)=0.49$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B=0.49$$

$$\cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta=0.49$$

$$\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta=0.49$$

$$\frac{1}{2}\left[1-\frac{\theta^2}{2!}+\cdots\right]-\frac{\sqrt{3}}{2}\left[\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right]=0.49$$

$$\frac{1}{2}-\frac{\sqrt{3}}{2}\theta=0.49$$

$$\theta=\frac{0.49\times 2}{1-\sqrt{3}}$$

$$=\frac{0.98}{1-1.732}=\frac{0.98}{0.732}$$

$$=1.3387$$

If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$ $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except when $q = 1$

Soln :-

This is an equation of degree 3. It has 3 roots namely x_1, x_2 & x_3.

$$S_1 = \frac{-\text{coeff.., of } x^2}{\text{coeff.., of } x^3} = -p$$

$$S_2 = \frac{\text{coeff.., of } x}{\text{coeff.., of } x^3} = q$$

$$S_3 = \frac{-\text{coeff.., of terms}}{\text{coeff.., of } x^3} = -p$$

$$\tan(x_1 + x_2 + x_3) = \frac{S_1 - S_3}{1 - S_2}$$

$$= \frac{-p - (-p)}{1 - q}$$

$$= \frac{-p + p}{1 - q} = \frac{0}{1 - q} = 0$$

$$x_1 + x_2 + x_3 = \tan^{-1}(0)$$

$$= n\pi \qquad n = 0, 1, 2 \ldots$$

$$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$$

Hence proved

When $x_1 = \tan^{-1}\alpha$

$x_2 = \tan^{-1}\beta$

$x_3 = \tan^{-1}\gamma$

Find the equation whose roots are

$\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$, $\tan 5\theta$.

Soln :-

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta + \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - nC_6 \tan^6\theta + \cdots}$$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

when $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$$\tan 5\theta = 0$$

$$0 = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2 + 5\tan^4\theta}$$

$$5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0 \quad \text{——— (1)}$$

Put $\tan\theta = x$

$$5x - 10x^3 + x^5 = 0$$

$$x^5 - 10x^3 + 5x = 0$$

$$x(x^4 - 10x^2 + 5) = 0$$

$$x^4 - 10x^2 + 5 = \frac{0}{x}$$

$$x^4 - 10x^2 + 5 = 0$$

Has roots $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$.

~ x ~

Expand $\tan 4\theta$ in terms of $\tan\theta$ & S.T. $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $\tan\frac{9\pi}{16}$, $\tan\frac{13\pi}{16}$ are roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

Soln :-

$$\tan n\theta = \frac{nC_1\tan\theta - nC_3\tan^3\theta + nC_5\tan^5\theta + \cdots}{1 - nC_2\tan^2\theta + nC_4\tan^4\theta - nC_6\tan^6\theta + \cdots}$$

$$\tan 4\theta = \frac{4C_1 \tan\theta - 4C_3 \tan^3\theta}{1 - 4C_2 \tan^2\theta + 4C_4 \tan^4\theta}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

When $\tan\theta = x$

$$= \frac{4x - 4x^3}{1 - 6x^2 + x^4} \quad \text{———} \quad (1)$$

Let θ denote any of the angles

$\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$4\theta = \left(n\pi + \frac{\pi}{4}\right) \quad n = 0, 1, 2, 3 \ldots$

$\tan 4\theta = \tan\left(n\pi + \frac{\pi}{4}\right)$

$\tan 4\theta = 1$

Sub in (1) $\Rightarrow 1 = \frac{4x - 4x^3}{1 - 6x^2 + x^4}$

$$1-6x^2+x^4 = 4x-4x^3$$

$$1-6x^2+x^4-4x+4x^3 = 0$$

$$x^4+4x^3-6x^2-4x+1 = 0$$

Has roots $\tan\frac{\pi}{16}$, $\tan\frac{5\pi}{16}$, $\tan\frac{9\pi}{16}$, $\tan\frac{13\pi}{16}$

~ x ~

[redacted]

I - B.sc Maths

[redacted]

Expansions of $\cos n\theta$:-

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta + \dots$$

Expansions of $\sin n\theta$:-

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + nC_5 \cos^{n-5}\theta \sin^5\theta \dots$$

Expansion of $\tan n\theta$:-

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - nC_6 \tan^6\theta \dots}$$

Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta \dots$$

$\boxed{n = 5}$

$$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta + 5C_4 \cos^{n-4}\theta \sin^4\theta$$

$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$\therefore$ Hence Proved.

Expand $\tan 5\theta$:-

$\boxed{n = 5}$

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta \ldots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta - nC_6 \tan^6\theta \ldots}$$

$$\tan 5\theta = \frac{5C_1 \tan\theta - 5C_3 \tan^3\theta + 5C_5 \tan^5\theta \ldots}{1 - 5C_2 \tan^2\theta + 5C_4 \tan^4\theta \ldots}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta \ldots}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

Q.7 $\frac{\cos 7\theta}{\cos\theta}$:-

$\boxed{n=7}$

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta + \cdots$$

$$\cos 7\theta = \cos^7\theta - 7C_2\cos^5\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta - 7C_6\cos\theta\sin^6\theta + \cdots$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta) - 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

$$\frac{\cos 7\theta}{\cos\theta} = \frac{64\cos^{\cancel{7}}\theta - 112\cos^{\cancel{5}}\theta + 56\cos^{\cancel{3}}\theta - 7\cos\theta}{\cos\cancel{\theta}}$$

$$= 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7.$$

S.T $\dfrac{\cos 5\theta}{\cos\theta}$:

$\boxed{n=5}$

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta \ldots$$

$$\cos 5\theta = \cos^5\theta - 5C_2\cos^3\theta\sin^2\theta + 5C_4\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$= 16\cos^5\theta - 20\cos^6\theta + 5\cos\theta$$

$$\frac{\cos 5\theta}{\cos\theta} = \frac{16\cos^5\theta - 20\cos^6\theta + 5\cancel{\cos\theta}}{\cancel{\cos\theta}}$$

$$= 16\cos^4\theta - 20\cos^5\theta + 5$$

Expand $\left[x+\frac{1}{x}\right]^2$:-

$$x = \cos\theta + i\sin\theta \longrightarrow (1)$$

$$\frac{1}{x} = \cos\theta - i\sin\theta \longrightarrow (2).$$

$$\left[x+\frac{1}{x}\right] = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

$$\left[x+\frac{1}{x}\right]^2 = [2\cos\theta]^2$$

$$= 4\cos^2\theta$$

Express $\cos^5\theta$ terms of cosine of multiple angles:-

$$x+\frac{1}{x} = 2\cos\theta.$$

$$a = x \;;\; b = \frac{1}{x} \;;\; n=5$$

$$\left(x+\frac{1}{x}\right)^5 = (2\cos\theta)^5$$

$$2^5\cos^5\theta = x^5 + 5C_1\, x^{5-1}\left(\frac{1}{x}\right) + 5C_2\, x^{5-2}\left(\frac{1}{x}\right)^2$$

$$+5C_3 x^{5-3}\left(\frac{1}{x}\right)^3 + 5C_4\, x^{5-4}\left(\frac{1}{x}\right)^4 + 5C_4\, x^{5-5}\left(\frac{1}{x}\right)^5$$

$$= x^5 + 5x^4 \times \frac{1}{x} + 10x^3 \times \frac{1}{x^2} + 10x^2 \times \frac{1}{x^3} + 5x \times \frac{1}{x^4} + \frac{1}{x^5}$$

$$= x^5 + 5x^3 + 10x + 10 \times \frac{1}{x} + 5 \times \frac{1}{x^3} + \frac{1}{x^5}$$

$$= \left[x^5 + \frac{1}{x^5}\right] + 5\left[x^3 + \frac{1}{x^3}\right] + 10\left[x + \frac{1}{x}\right]$$

$$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$$

$$2^5\cos^5\theta = 2(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

$$\cos^5\theta = \frac{1}{2^4}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

Expand $\sin^6\theta$:

$$x - \frac{1}{x} = 2i\sin\theta.$$

$$a = x\ ;\ b = \frac{1}{x}\ ;\ n = 6.$$

$$\left[x - \frac{1}{x}\right]^6 = (2i\sin\theta)^6$$

$$2^6 i^6 \sin^6\theta = x^6 - 6C_1 x^{6-1}\left(\frac{1}{x}\right) + 6C_2 x^{6-2}\left(\frac{1}{x}\right)^2 - 6C_3 x^{6-3}\left(\frac{1}{x}\right)^3 + 6C_4 x^{6-4}\left(\frac{1}{x}\right)^4 - 6C_5 x^{6-5}\left(\frac{1}{x}\right)^5 + 6C_6 x^{6-6}\left(\frac{1}{x}\right)^6$$

$$= x^6 - 6x^5 \times \frac{1}{x} + 15x^4 \times \frac{1}{x^2} - 20x^3 \times \frac{1}{x^3} + 15x^2 \times \frac{1}{x^4} - 6x \times \frac{1}{x^5} + (1) \times \frac{1}{x^6}$$

$$= x^6 - 6x^4 + 15x^2 - 20 + 15 \times \frac{1}{x^2} - 6 \times \frac{1}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$2^6 i^6 \sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$= 2(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$$

$$\sin^6\theta = \frac{-1}{2^5}\left[\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right]$$

Expansions of $\sin\theta$:-

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Expansions of $\cos\theta$:-

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

Expansions of $\tan\theta$:-

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Evaluate :- $\lim_{\theta \to 0} \left[\frac{\theta - \sin\theta}{\theta^3} \right]$

$$\lim_{\theta \to 0} \left[\frac{\theta - \sin\theta}{\theta^3} \right] = \lim_{\theta \to 0} \frac{\theta - \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \right]}{\theta^3}$$

$$= \lim_{\theta \to 0} \frac{\cancel{\theta} - \cancel{\theta} + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \cdots}{\theta^3}$$

$$= \lim_{\theta \to 0} \frac{\cancel{\theta^3} \left[\frac{1}{3!} - \frac{\theta^2}{5!} + \cdots \right]}{\cancel{\theta^3}}$$

$$= \frac{1}{3!} = \frac{1}{1\times2\times3} = \frac{1}{6}.$$

If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$ P.T $\theta = 2^{0}$

W.K.T., $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$

$$\frac{\sin\theta}{\theta} = \theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^3}{5!} + \dots\right]$$

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^3}{5!} + \dots$$

$\frac{2165}{2166} = 1 - \frac{\theta^2}{6}$ [neglecting high powers of θ]

$$1 - \frac{\theta^2}{6} = 1 - \frac{1}{2166}$$

$$\frac{\theta^2}{6} = \frac{1}{2166}$$

$$\theta^2 = \frac{6}{2166}$$

$$\theta = \frac{1}{361}$$

$$(\theta^2)^{1/2} = \left[\frac{1}{361}\right]^{1/2}$$

$\theta = \left[\frac{1}{19^2}\right]^{1/2}$

$\theta = \frac{1}{19}$ radians.

$\theta = \frac{1}{19} \times \frac{180}{\pi}$ deg

$\theta = \frac{1}{19} \times \frac{180}{22/7}$

$\theta = \frac{1}{19} \times \frac{180}{22} \times 7$

$\theta = \frac{1260}{418}$

$\theta = 3^0$ nearly

$\therefore$ Hence Proved.

Find the equation whose roots are $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$ and $\tan\frac{4\pi}{5}$.

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + nC_5 \tan^5\theta \ldots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta \ldots}$$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1-10\tan^2\theta + 5\tan^4\theta} \ldots$$

when $\theta = 0, \pi/5, 2\pi/5, 3\pi/5, 4\pi/5 \ldots$

$\tan 5\theta = 0$

$$0 = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1-10\tan^2\theta + 5\tan^4\theta} \ldots$$

$5\tan\theta - 10\tan^3\theta + \tan^5\theta = 0 \longrightarrow (1)$

put $\tan\theta = x$ in (1)

$5x - 10x^3 + x^5 = 0$

$x(5 - 10x^2 + x^4) = 0$

$x^4 - 10x^2 + 5 = 0$

has roots $\tan \pi/5$, $\tan 2\pi/5$, $\tan 3\pi/5$ and $\tan 4\pi/5$

$\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - \sinh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2$$

$$= \frac{(e^{x}+e^{-x})^2-(e^{x}-e^{-x})^2}{4} \qquad [\because e^0=1]$$

$$= \frac{e^{2x}+e^{-2x}+2e^{x}e^{-x}-[e^{2x}+e^{-2x}-2e^{x}e^{-x}]}{4}$$

$$= \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2-\cancel{e^{+2x}}-\cancel{e^{-2x}}+2}{4}$$

$$= \frac{4}{4} = 1$$

$\operatorname{sech}^2 x + \tanh^2 x = 1$

$$\operatorname{sech}^2 x + \tanh^2 x = \left[\frac{2}{e^{x}+e^{-x}}\right]^2 + \left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]^2$$

$$= \frac{4+(e^{x}-e^{-x})^2}{(e^{x}+e^{-x})^2}$$

$$= \frac{4+e^{2x}+e^{-2x}-2e^{x}e^{-x}}{e^{2x}+e^{-2x}+2e^{x}e^{-x}}$$

$$= \frac{\cancel{e^{2x}}+\cancel{e^{-2x}}+2}{\cancel{e^{2x}}+\cancel{e^{-2x}}+2} = \frac{2}{2} = 1$$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$A = ix \; ; \; B = iy.$

$\sinh(x \pm y) = \frac{1}{i} \sin i(\underset{A}{x} \pm \underset{B}{y})$ $\left[\because \sin ix = i \sinh x \right.$
$\left. \cos ix = \cosh x \right]$

$= \frac{1}{i} \sin(ix \pm iy)$

$= \frac{1}{i}(\sin ix \cos iy \pm \cos ix \sin iy)$

$= \frac{1}{i}(i \sinh x \cosh y \pm \cosh x \, i \sinh y)$

$= \frac{\cancel{i}}{\cancel{i}}(\sinh x \cosh y \pm \cosh x \sinh y)$

LHS = RHS.

$\therefore$ Hence Proved.

$\sinh 2x = 2 \sinh x \cosh x.$

$= \frac{1}{i} \sin i 2x$

$= \frac{1}{i} \sin 2(ix)$

$= \frac{1}{i} 2 \sin ix \cos ix$

$= \frac{1}{i}(2i\sinh x \cosh x)$

$= 2\sinh x \cosh x$.

$1-\tanh^2 x = \operatorname{sech}^2 x$.

$1+\tan^2\theta = \sec^2\theta$

$\theta = ix$.

$1+\tan^2(ix) = \sec^2(ix)$

$1+(i\tanh x)^2 = (\operatorname{sech} x)^2$

$1-\tanh^2 x = \operatorname{sech}^2 x$.

$\cos ix = \cosh x$.

$\frac{1}{\cos ix} = \frac{1}{\cosh x}$.

$\sec ix = \operatorname{sech} x$.

P.T $\sinh 3x = 3\sinh x + 4\sinh^3 x$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\boxed{\theta = ix}$

$\sin 3(ix) = 3\sin(ix) - 4\sin^3(ix)$

$\sin i(3x) = 3(i\sinh x) - 4(i\sinh x)^3$

$i\sinh 3x = 3(i\sinh x) - 4(i^3\sinh^3 x)$

$$i \sinh 3x = 3i \sinh x + 4i \sinh^3 x$$

$$i \sinh 3x = i\,[3 \sinh x + 4 \sinh^3 x]$$

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$$

//

$$\frac{1+\cos 2x}{2} = \cos^2 x$$

$$\boxed{x = ix}$$

$$\frac{1+\cos 2(ix)}{2} = \cos^2(ix)$$

$$\frac{1+\cos i(2x)}{2} = \cos^2(ix)$$

$$\frac{1+\cosh(2x)}{2} = \cosh^2 x$$

$$\frac{1-\cos 2x}{2} = \sin^2 x$$

$$\boxed{x = ix}$$

$$\frac{1-\cos 2(ix)}{2} = \sin^2(ix)$$

$\because \cos ix = \cosh x$.

$\cos 2(ix) = \cos i(2x)$

$= \cosh(2x)$

$$\frac{1-\cos i(2x)}{2} = \sin^2(ix)$$

$$\frac{1-\cosh 2x}{2} = (i\sinh x)^2$$

$[i^2 = -1]$

$$\frac{1-\cosh 2x}{2} = -\sinh^2 x.$$

$$\frac{\cancel{-}(\cosh 2x-1)}{2} = \cancel{-}\sinh^2 x$$

$$\frac{\cosh 2x-1}{2} = \sinh^2 x.$$

Seperate the following real and imaginary part.

$\sin(x+iy)$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$.

$\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$.

$= \sin x\cosh y + \cos x\, i\sinh y$.

Real part $= \sin x \cosh y$.

Imaginary part $= \cos x \sinh y$.

$\cos(x+iy)$ *

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - \sin x \, i \sinh y$$

$R \cdot P = \cos x \cosh y$.

$I \cdot P = -\sin x \sinh y$.

$\tan(x+iy)$

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$x = x+iy$; $y = x-iy$.

$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{2\sin A\cos B}{2\cos A\cos B}$$

$$= \frac{\sin(x+iy+x-iy)+\sin[x+iy-(x-iy)]}{\cos(x+iy+x-iy)+\cos[x+iy-(x-iy)]}$$

$$= \frac{\sin 2x+\sin 2iy}{\cos 2x+\cos 2iy} = \frac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$$

$$R.P = \frac{\sin 2x}{\cos 2x+\cosh 2y} \; ; \; I.P = \frac{\sinh 2y}{\cos 2x+\cosh 2y}.$$

If $\sin(A+iB) = x+iy$; P.T $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$.

$\sin(A+iB) = x+iy$.

$\sin A\cos iB + \cos A\sin iB = x+iy$.

$\sin A\cosh B + \cos A\, i\sinh B = x+iy$.

$x = \sin A\cosh B$

$y = \cos A\sinh B$.

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \frac{(\sin A \cosh B)^2}{\sin^2 A} - \frac{(\cos A \sinh B)^2}{\cos^2 A}$$

$$= \frac{\sin^2 A \cosh^2 B}{\sin^2 A} - \frac{\cos^2 A \sinh^2 B}{\cos^2 A}$$

$$= \cosh^2 B - \sinh^2 B$$

$$= 1$$

$$= RHS$$

$\therefore$ Hence Proved.

P.T. $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$.

$x = \sin A \cosh B$

$y = \cos A \sinh B$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \frac{(\sin A \cosh B)^2}{\cosh^2 B} + \frac{(\cos A \sinh B)^2}{\sinh^2 B}$$

$$= \frac{\sin^2 A \cancel{\cosh^2 B}}{\cancel{\cosh^2 B}} + \frac{\cos^2 A \cancel{\sinh^2 B}}{\sinh^2 B}.$$

$= \sin^2 A + \cos^2 A$

$= 1$ //

$\therefore$ Hence Proved.

Seperate the real and imaginary part of $\sinh(x+iy)$

Let $\alpha + i\beta = \sinh(x+iy)$

$= \frac{1}{i} \sin i(x+iy)$

× ly and ÷ by i

$= \frac{i}{i \times i} \sin(xi - y)$

$A = xi \; ; \; B = y$

$= -i(\sin ix \cos y - \cos ix \sin y)$

$= -i(i \sinh x \cos y - \cosh x \sin y)$

$= \sinh x \cos y + i \cosh x \sin y$.

$R.P = \sinh x \cos y$.

$I.P = \cosh x \sin y$.

If $\tan(A+iB) = x+iy$ P.T $x^2+y^2+2x\cot 2A=1$

$$\tan 2A = \frac{2x}{1-x^2-y^2}$$

$$\frac{1}{\cot 2A} = \frac{2x}{1-x^2-y^2}$$

$$1-x^2-y^2 = 2x\cot 2A$$

$$1 = x^2+y^2+2x\cot 2A \;//$$

Separate into real and imaginary of $\cosh(1+i)$

$$\cosh(1+i) = \cos i(1+i)$$

$$= \cos(i-1)$$

$$= \cos i\cos 1 + \sin i\sin 1$$

$$= \cosh\cos 1 + i\sinh\sin 1$$

$R.P = \cosh\cos 1$

$I.P = \sinh\sin 1$

$$\frac{(\cosh x \cos y)^2}{\cos^2 y} - \frac{(\sinh x \sin y)^2}{\sin^2 y}$$

$$\frac{\cosh^2 x \cos^2 y}{\cos^2 y} - \frac{\sinh^2 x \sin^2 y}{\sin^2 y}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$\therefore$ Hence Proved.

If $\cos^{-1}(\alpha + i\beta) = \theta + i\phi$ P.T

$$\alpha^2 \operatorname{sech}^2\phi + \beta^2 \operatorname{cosech}^2\phi = 1$$

$$\cos^{-1}(\alpha + i\beta) = \theta + i\phi$$

$$\alpha + i\beta = \cos(\theta + i\phi)$$

$$= \cos\theta \cos i\phi - \sin\theta \sin i\phi$$

$$= \cos\theta \cosh\phi - i\sin\theta \sinh\phi$$

$\alpha = \cos\theta \cosh\phi \qquad \beta = -\sin\theta \sinh\phi$

$\alpha^2 = \cos^2\theta \cosh^2\phi \qquad \beta^2 = \sin^2\theta \sinh^2\phi$

$$\frac{\alpha^2}{\cosh^2\phi} = \cos^2\theta \longrightarrow ① \qquad \frac{\beta^2}{\sinh^2\phi} = \sin^2\theta \longrightarrow ②$$

$$① + ② \Rightarrow \frac{\alpha^2}{\cosh^2\phi} + \frac{\beta^2}{\sinh^2\phi} = \cos^2\theta + \sin^2\theta$$

$$\alpha^2 \operatorname{sech}^2\phi + \beta^2 \operatorname{cosech}^2\phi = 1$$

$\therefore$ Hence Proved.

If $\cos(x+iy) = \cos\theta + i\sin\theta$.

S.T $\cos 2x + \cosh 2y = 2$.

$$\cos(x+iy) = \cos\theta + i\sin\theta.$$

$$\cos(x-iy) = \cos\theta - i\sin\theta.$$

$$2\cos(x+iy)\cos(x-iy) = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$A = x+iy \;;\; B = x-iy.$$

$$\cos[x+iy+(x-iy)] + \cos[x+iy-(x-iy)] = 2\times 1$$

$$\cos 2x + \cos 2iy = 2$$

$\cos 2x + \cosh 2y = 2$ //

$\therefore$ Hence Proved.

S.T $\log(1+i\tan\alpha) = \log(\sec\alpha) + i\alpha$.

$x = 1$, $y = \tan\alpha$.

$$\log(1+i\tan\alpha) = \frac{1}{2}\log(1^2+\tan^2\alpha) + i\tan^{-1}\left[\frac{\tan\alpha}{1}\right]$$

$$= \frac{1}{2}\log \sec^2\alpha + i\alpha$$

$$= \log(\sec^2\alpha)^{1/2} + i\alpha$$

$$= \log \sec\alpha + i\alpha.$$

$\therefore$ Hence Proved //.

Find the value of $\operatorname{Log} i(1+i)$

$$\operatorname{Log}(x+iy) = 2\pi n i + \log(x+iy)$$

$$\operatorname{Log} i = 2\pi n i + \log i$$

$= 2\pi n i + \log(\cos \pi/2 + i \sin \pi/2)$

$= 2\pi n i + \log(e^{i\pi/2})$

$= 2\pi n i + i\pi/2$

$= \frac{\pi i}{2}(4n+1)$

$\mathrm{Log}\, 3i$.

$\mathrm{Log}(x+iy) = 2\pi n i + \log(x+iy)$

$\mathrm{Log}\, i = 2\pi n i + \log 3i$

$= 2\pi n i + \log 3(\cos \pi/2 + i \sin \pi/2)$

$= 2\pi n i + \log 3(e^{i\pi/2})$

$= 2\pi n i + \log 3 + \log e^{i\pi/2}$

$= 2\pi n i + \log 3 + i\pi/2$

$= \log 3 + i\pi/2\,(4n+1)$

$2\pi n i + i\pi/2$

$\frac{4\pi n i}{2}$

$= i\pi/2\,(4n+1)$

$\log \sqrt{i}$

$$\log(x+iy) = 2\pi n i + \log \sqrt{i}$$

$$= 2\pi n i + \log \sqrt{\cos \pi/2 + i \sin \pi/2}$$

$$= 2\pi n i + \log \sqrt{e^{i\pi/2}}$$

$$= 2\pi n i + \log (e^{i\pi/2})^{1/2}$$

$$= 2\pi n i + \log e^{i\pi/4}$$

$$= 2\pi n i + i\pi/4 = \pi i\left(2n + \frac{1}{4}\right) = \frac{\pi i}{4}(8n+1)$$

Trigonometry.

10 - Marks

$$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

Soln:

$$\cos n\theta = \cos^n\theta - nc_2\cos^{n-2}\theta\sin^2\theta + nc_4\cos^{n-4}\sin^4\theta - nc_6\cos^{n-6}\theta\sin^6\theta$$

Here $n = 7$

$$\cos 7\theta = \cos^7\theta - 7c_2\cos^5\theta\sin^2\theta + 7c_4\cos^3\theta\sin^4\theta - 7c_6\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta[1-\cos^2\theta] + 35\cos^3\theta[1-2\cos^2\theta + \cos^4\theta] - 7\cos[1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta]$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

$$\frac{\cos 7\theta}{\cos\theta} = \frac{64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta}{\cos\theta}$$

$$= 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7$$

$$\therefore \frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 112\cos^4\theta + 56\cos^2\theta - 7 \,//$$

Hence proved //

$\frac{\sin 5\theta}{\sin\theta}$

$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta \ldots$

Soln:

Here $n = 5$

$\sin 5\theta = 5c_1 \cos^4\theta \sin\theta - 5c_3 \cos^2\theta \sin^3\theta + 5c_5 \cos^0 \sin^5\theta$

$= 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + (1) \sin^5\theta$

$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta [1 - \sin^2\theta + \sin^5\theta]$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$\frac{\sin 5\theta}{\sin\theta} = \frac{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}{\sin\theta}$

$= 16\sin^4\theta - 20\sin^2\theta + 5.$

$\frac{\sin 5\theta}{\sin\theta} = 16\sin^4\theta - 20\sin^2\theta + 5$ //

$\frac{\cos 5\theta}{\cos\theta}$

Soln:

$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \ldots$

Here $n = 5$

$\cos 5\theta = \cos^5\theta - 5c_2 \cos^3\theta \sin^2\theta + 5c_4 \cos^1\theta \sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos \sin^4\theta$

$$= \cos^5\theta - 10\cos^3\theta\,[1-\cos^2\theta] + 5\cos\theta[1 - 2\cos^2\theta + \cos^4\theta]$$

$$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

$$\frac{\cos 5\theta}{\cos\theta} = \frac{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}{\cos\theta}$$

$$= 16\cos^4\theta - 20\cos^2\theta + 5 \,//$$

$$\frac{\cos 5\theta}{\cos\theta} = 16\cos^4\theta - 20\cos^2\theta + 5$$

Hence proved//

$$\frac{\sin 8\theta}{\sin\theta}$$

Soln:

$$\sin n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta - nc_7 \cos^{n-7}\theta \sin^7\theta$$

Here $n = 8$

$$\sin 8\theta = 8c_1 \cos^7\theta \sin\theta - 8c_3 \cos^5\theta \sin^3\theta + 8c_5 \cos^3\theta \sin^5\theta - 8c_7 \cos\theta \sin^7\theta$$

Here

$$\frac{\sin 8\theta}{\sin\theta} = \frac{8c_1 \cos^7\theta \sin\theta - 8c_3 \cos^5\theta \sin^3\theta + 8c_5 \cos^3\theta \sin^5\theta - 8c_7 \cos\theta \sin^7\theta}{\sin\theta}$$

$$= 8\cos^7\theta - 56\cos^5\theta\,[1-\cos^2\theta] + 56\cos^3\theta\,[1 - 2\cos^2\theta + \cos^4\theta] - 8\cos\theta[1 - 3\cos^2\theta + 3\cos^4\theta + \cos^6\theta]$$

$$= 8\cos^7\theta - 56\cos^5\theta + 56\cos^7\theta + 56\cos^3\theta - 112\cos^5\theta$$

$= 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta$

$\frac{\sin 8\theta}{\sin\theta} = 128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta$ //

$\cos 9\theta$

Soln :

$\cos n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\sin^4\theta$

$-nc_6 \cos^{n-6}\theta \sin^6\theta + nc_8 \cos^{n-8}\theta \sin^8\theta$

Here $n = 9$

$\cos 9\theta = \cos^9\theta - 9c_2 \cos^7\theta \sin^2\theta + 9c_4 \cos^5\theta \sin^4\theta$

$-9c_6 \cos^3\theta \sin^6\theta + 9c_8 \cos\theta \sin^8\theta$

$= \cos^9\theta - 36\cos^7\theta \sin^2\theta + 126\cos^5\theta \sin^4\theta - 84\cos^3\theta$

$\sin^6\theta + 9\cos\theta \sin^8\theta$

$= \cos^9\theta - 36\cos^7\theta[1-\cos^2\theta] + 126\cos^5\theta[1-2\cos^2\theta$

$+\cos^4\theta] - 84\cos^3\theta[1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta]$

$+9\cos\theta[1-4\cos^2\theta + 6\cos^4\theta - 4\cos^6\theta + \cos^8\theta]$

$= \cos^9\theta - 36\cos^7\theta + 36\cos^9\theta + 126\cos^5\theta - 252$

$\cos^7\theta + 126\cos^9\theta - 84\cos^3\theta + 252\cos^5\theta$

$-252\cos^7\theta + 84\cos^9\theta + 9\cos\theta - 36\cos^3\theta$

$+54\cos^5\theta - 36\cos^7\theta + 9\cos^9\theta$

Answer:

$\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta$

$+9\cos\theta$ //

Expand $\cos^5\theta \sin^4\theta$ in terms of cosine multiple angle:

Soln:

$$\left[x+\frac{1}{x}\right]\left[x-\frac{1}{x}\right] = [2\cos\theta][2i\sin\theta]$$

$$[2\cos\theta]^5[2i\sin\theta]^4 = \left[x+\frac{1}{x}\right]^5\left[x-\frac{1}{x}\right]^4$$

$$= \left[x+\frac{1}{x}\right]\left[x+\frac{1}{x}\right]^4\left[x-\frac{1}{x}\right]^4$$

$$= \left[x+\frac{1}{x}\right]\left[x^2-\frac{1}{x^2}\right]^4$$

$$= \left[x+\frac{1}{x}\right]\left[(x^2)^4 - 4(x^2)^3 \times \frac{1}{x^2} + 6(x^2)^2\times\left(\frac{1}{x^2}\right)^2 - 4(x^2) \times\left[\frac{1}{x^2}\right]^3 + \left(\frac{1}{x^2}\right)^4\right]$$

$$= \left[x+\frac{1}{x}\right]\left[x^8 - 4x^4 + 6 - 4\times\frac{1}{x} + \frac{1}{x^8}\right]$$

$$= x^9 - 4x^5 + 6x - 6\frac{1}{x^3} + \frac{1}{x^7} + x^7 - 4x^3 + 6\frac{1}{x} - 4\frac{1}{x^5} + \frac{1}{x^9}$$

$$= \left[x^9+\frac{1}{x^9}\right] - 4\left[x^5+\frac{1}{x^5}\right] + 6\left[x+\frac{1}{x}\right] - 4\left[x^3+\frac{1}{x^3}\right] + \left[x^7+\frac{1}{x^7}\right]$$

$$2^5\cos^5\theta \cdot 2^4 i^4 \sin^4\theta$$

$$= 2\cos 9\theta - 4[2\cos 5\theta] + 6[2\cos\theta] - 4[2\cos 3\theta] + [2\cos 7\theta]$$

$$= 2[\cos 9\theta - 4\cos 5\theta + 6\cos\theta - 4\cos 3\theta + \cos 7\theta]$$

$$\cos^5\theta\sin^4\theta = \frac{2}{(1)\ 2^9}[\cos 9\theta - 4\cos 5\theta + 6\cos\theta - 4\cos 3\theta + \cos 7\theta]$$

Answer:

$$\cos^5\theta\sin^4\theta = \frac{1}{2^8}[\cos 9\theta - 4\cos 5\theta + 6\cos\theta - 4\cos 3\theta + \cos 7\theta]$$

Expand $\cos^4\theta\sin^3\theta$ in terms of sine of multiple angle.

Soln:

$$\left[x+\frac{1}{x}\right]\left[x-\frac{1}{x}\right] = [2\cos\theta][2i\sin\theta]$$

$$[2\cos\theta]^4[2i\sin\theta]^3 = \left[x+\frac{1}{x}\right]^4\left[x-\frac{1}{x}\right]^3$$

$$= \left[x+\frac{1}{x}\right]\left[x+\frac{1}{x}\right]^3\left[x-\frac{1}{x}\right]^3$$

$$= \left[x+\frac{1}{x}\right]\left[x^2-\frac{1}{x^2}\right]^3$$

$$= \left[x+\frac{1}{x}\right]\left[a^n - nc_1 a^{n-1}b + nc_2 a^{n-2}b^2 - nc_3 a^{n-3}b^3 + \cdots\right]$$

$a = x^2 \quad b = \frac{1}{x^2} \quad n = 3$

$$= \left[x+\frac{1}{x}\right]\left[(x^2)^3 - 3c_1(x^2)^2\left(\frac{1}{x^2}\right) + 3c_2(x^2)\left(\frac{1}{x^2}\right)^2 - 3c_3(x^2)^0\left(\frac{1}{x^2}\right)^3\right]$$

$$= \left[x+\frac{1}{x}\right]\left[x^6 - 3x^2 + 3\frac{1}{x^2} - \frac{1}{x^6}\right]$$

$$= x^7 - 3x^3 + 3\frac{1}{x} - \frac{1}{x^5} + x^5 - 3x + 3\frac{1}{x^3} - \frac{1}{x^7}$$

$$= \left[x^7 - \frac{1}{x^7}\right] - 3\left[x^3 - \frac{1}{x^3}\right] - 3\left[x - \frac{1}{x}\right] + \left[x^5 - \frac{1}{x^5}\right]$$

$$= [2i\sin 7\theta] - 3[2i\sin 3\theta] - 3[2i\sin\theta] + [2i\sin 5\theta]$$

$$2^4\cos^4\theta \cdot 2^3 i^3 \sin^3\theta$$

$$= 2i\sin 7\theta - 3[2i\sin 3\theta] - 3[2i\sin\theta] + [2i\sin 5\theta]$$

$$\cos^4\theta\sin^3\theta = \frac{2i}{(-i)(2^7)}[\sin 7\theta - 3\sin 3\theta - 3\sin\theta + \sin 5\theta]$$

$$= -\frac{1}{2^6}[\sin^7\theta - 3\sin 3\theta - 3\sin\theta + \sin 5\theta]$$

Answer:

$$\therefore \cos^4\theta\sin^3\theta = -\frac{1}{2^6}[\sin^7\theta - 3\sin 3\theta - 3\sin\theta + \sin 5\theta] //$$

$$\cos^8\theta + \sin^8\theta = \frac{1}{64}[\cos 8\theta + 28\cos 4\theta + 35]$$

Soln: I Part

$$[2\cos\theta]^8 = \left[x + \frac{1}{x}\right]^8$$

$$(a+b)^n = a^n + nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 + nc_3 a^{n-3} b^3 + nc_4 a^{n-4} b^4 + nc_5 a^{n-5} b^5 + nc_6 a^{n-6} b^6 + nc_7 a^{n-7} b^7 + nc_8 a^{n-8} b^8$$

$$\left[a = x \quad b = \frac{1}{x} \quad n = 8\right]$$

$$= x^8 + 8x^7\left(\frac{1}{x}\right) + 28x^6\left(\frac{1}{x}\right)^2 + 56x^5\left(\frac{1}{x}\right)^3 + 70x^4\left(\frac{1}{x}\right)^4 + 56x^3\left(\frac{1}{x}\right)^5 + 28x^2\left(\frac{1}{x}\right)^6 + 8x\left(\frac{1}{x}\right)^7 + \left(\frac{1}{x}\right)^8$$

$$= \left[x^8 + \frac{1}{x^8}\right] + 8\left[x^6 + \frac{1}{x^6}\right] + 28\left[x^4 + \frac{1}{x^4}\right] + 56\left[x^2 + \frac{1}{x^2}\right] + 70$$

$$2^8 \cos^8\theta = 2\cos 8\theta + 8[2\cos 6\theta] + 28[2\cos 4\theta] + 56[2\cos 2\theta]$$

$$\cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$$ //

(ii) II Part

$\sin^8\theta$

Soln:

$$[2i\sin\theta]^8 = \left[x - \frac{1}{x}\right]^8$$

$$2^8 i^8 \sin^8\theta = \left[x - \frac{1}{x}\right]^8$$

$$(a-b)^n = a^n - nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 - nc_3 a^{n-3} b^3 + nc_4 a^{n-4} b^4 - nc_5 a^{n-5} b^5 + nc_6 a^{n-6} b^6 - nc_7 a^{n-7} b^7 + nc_8 a^{n-8} b^8$$

$$\left[a = x \quad b = \frac{1}{x} \quad n = 8\right]$$

$$= x^8 - 8x^7\left(\frac{1}{x}\right) + 28x^6\left(\frac{1}{x}\right)^2 - 56(x^5)\left(\frac{1}{x}\right)^3 + 70x^4\left(\frac{1}{x}\right)^4 - 56x^3\left(\frac{1}{x}\right)^5 + 28x^2\left(\frac{1}{x}\right)^6 - 8x\left(\frac{1}{x}\right)^7 + \left(\frac{1}{x}\right)^8$$

$$= \left[x^8 + \frac{1}{x^8}\right] - 8\left[x^6 + \frac{1}{x^6}\right] + 28\left[x^4 + \frac{1}{x^4}\right] - 56\left[x^2 + \frac{1}{x^2}\right] + 70$$

$$= 2\cos 8\theta - 8[2\cos 6\theta] + 28[2\cos 4\theta] - 56[2\cos 2\theta] + 70$$

$$\sin^8\theta = \frac{2}{2^8}[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$$

$$\sin^8\theta = \frac{1}{2^7}\left[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35\right]$$

Here $\cos^8\theta + \sin^8\theta$

$$= \frac{1}{2^7}\left[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\right]$$

$$+ \frac{1}{2^7}\left[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35\right]$$

$$= \frac{2}{2^7}\left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$= \frac{1}{2^6}\left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$= \frac{1}{64}\left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$\therefore \cos^8\theta + \sin^8\theta = \frac{1}{64}\left[\cos 8\theta + 28\cos 4\theta + 35\right]$$

$$\lim_{\theta\to 0}\left[\frac{3\sin\theta - \sin 3\theta}{\theta - \sin\theta}\right] = 24$$

Soln:

$$= \frac{3\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right] - \left[3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!}\right]}{\left[\theta - \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right]}$$

$$= \frac{3\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right] - \left[3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!}\right]}{\theta^3\left[\frac{1}{3!} + \frac{\theta^2}{5!}\right]}$$

$$= \lim_{\theta \to 0} \frac{3\theta - \frac{3\theta^3}{3!} + \frac{3\theta^5}{5!} - 3\theta + \frac{(3\theta)^3}{3!} - \frac{(3\theta)^5}{5!}}{\theta^3 \left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{3\theta - \frac{3\theta^3}{3!} + \frac{3\theta^5}{5!} - 3\theta + \frac{27\theta^3}{3!} - \frac{243\theta^5}{5!}}{\theta^3 \left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{-\frac{3\theta^3}{3!} + \frac{3\theta^5}{5!} \cdots + \frac{27\theta^3}{3!} - \frac{243\theta^5}{5!}}{\theta^3 \left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{\theta^3 \left[-\frac{3}{3!} + \frac{3\theta^2}{5!} \cdots + \frac{27}{3!} - \frac{243\theta^2}{5!} \right]}{\theta^3 \left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{3\theta^3 \left[-\frac{1}{3!} + \frac{\theta^2}{5!} + \frac{9}{3!} - \frac{81\theta^2}{5!} \right]}{\theta^3 \left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{3 \left[-\frac{1}{3!} + \frac{\theta^2}{5!} \cdots + \frac{9}{3!} - \frac{81\theta^2}{5!} \right]}{\left[\frac{1}{3!} + \frac{\theta^2}{5!} \right]}$$

$$= \lim_{\theta \to 0} \frac{3\left[-\frac{1}{3\times 2} + \frac{\theta^2}{5!} \cdots + \frac{9}{3\times 2} - \frac{81\theta^2}{5!}\right]}{\left[\frac{1}{3!} + \frac{\theta^2}{5!}\right]}$$

$$= \frac{3\left[-\frac{1}{6} + \frac{9}{6}\right]}{\frac{1}{6}} = 3 \times \frac{8}{6} \times \frac{6}{1}$$

$$= 24 //$$

$$\therefore \lim_{\theta \to 0} \left[\frac{3\sin\theta - \sin 3\theta}{\theta - \sin\theta}\right] = 24 //$$

Hence proved//

Evaluate $\lim_{\theta \to 0} \left[\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}\right]$

Soln:

$$= \lim_{\theta \to 0} \left[\frac{\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} + 1}\right]$$

$$= \lim_{\theta \to 0} \left[\frac{\frac{\sin\theta + 1 - \cos\theta}{\cos\theta}}{\frac{\sin\theta - 1 + \cos\theta}{\cos\theta}}\right]$$

$$= \lim_{\theta \to 0} \left[\frac{\sin\theta + 1 - \cos\theta}{\cos\theta} \times \frac{\cos\theta}{\sin\theta - 1 + \cos\theta} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\sin\theta + 1 - \cos\theta}{\sin\theta - 1 + \cos\theta} \right]$$

$$= \lim_{\theta \to 0} \frac{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right] + 1 - \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right]}{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right] - 1 + \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right]}$$

$$= \lim_{\theta \to 0} \left[\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots + 1 - 1 + \frac{\theta^2}{2!} - \frac{\theta^4}{4!}}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots - 1 + 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} \cdots}{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots} \right]$$

$$= \lim_{\theta \to 0} \frac{\theta}{\theta} \left[\frac{1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta}{2!} + \frac{\theta^3}{4!}}{1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta}{2!} + \frac{\theta^3}{4!}} \right]$$

$$= \lim_{\theta \to 0} \left[\frac{1 - \frac{\theta^2}{3!} - \frac{\theta}{2!}}{1 - \frac{\theta^2}{3!} - \frac{\theta}{2!}} \right]$$

$$= \left[\frac{1-0}{1-0}\right]$$

$$= \frac{1}{1} = 1$$

$$\therefore \lim_{\theta \to 0} \left[\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}\right] = 1$$

Evaluate $\displaystyle \lim_{\theta \to 0} \frac{n \sin\theta - \sin n\theta}{\theta(\cos\theta - \cos n\theta)}$

Soln:

$$\lim_{\theta \to 0} = \frac{n\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right] - \left[n\theta - \frac{(n\theta)^3}{3!} + \frac{(n\theta)^5}{5!} + \cdots\right]}{\theta\left[\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right] - \left[1 - \frac{(n\theta)^2}{2!} + \frac{(n\theta)^4}{4!} \cdots\right]\right]}$$

$$= \lim_{\theta \to 0} \frac{n\theta - \frac{n\theta^3}{3!} + \frac{n\theta^5}{5!} - n\theta + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!}}{\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \left[1 - \frac{n^2\theta^2}{2!} + \frac{n^4\theta^4}{4!}\right]\right]}$$

$$= \lim_{\theta \to 0} \frac{-\frac{n\theta^3}{3!} + \frac{n\theta^5}{5!} + \cdots + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!} \cdots}{\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots - 1 + \frac{n^2\theta^2}{2!} - \frac{n^4\theta^4}{4!}\right]}$$

$$= \lim_{\theta \to 0} \frac{\frac{\theta^3}{3!}[n^3-n] + \frac{\theta^5}{5!}[n-n^5]+\cdots}{\theta\left[-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\cdots+\frac{n\theta^2}{2!}-\frac{n\theta^4}{4!}\right]}$$

$$= \lim_{\theta \to 0} \frac{\frac{\theta^3}{3!}[n^3-n] + \frac{\theta^5}{5!}[n-n^5]+\cdots}{\theta\left[\frac{\theta^2}{2!}[n^2-1]+\frac{\theta^4}{4!}[1-n^4]+\cdots\right.}$$

$$= \lim_{\theta \to 0} \frac{\frac{\theta^3}{3!}[n^3-n] + \frac{\theta^5}{5!}[n-n^5]+\cdots}{\theta.\theta^2\left[\frac{1}{2!}[n^2-1]+\frac{\theta^2}{4!}[1-n^4]+\cdots\right]}$$

$$= \lim_{\theta \to 0} \frac{\frac{\theta^3}{3!}[n^3-n] + \frac{\theta^5}{5!}[n-n^5]+\cdots}{\theta^3\left[\frac{1}{2!}[n^2-1]+\frac{\theta^2}{4!}[1-n^4]+\cdots\right.}$$

$$= \lim_{\theta \to 0} \frac{\left[\theta^3\left[\frac{1}{3!}[n^3-n]+\frac{\theta^2}{5!}[n-n^5]\right]\right]}{\theta^3\left[\frac{1}{2!}[n^2-1]+\frac{\theta^2}{4!}[1-n^4]+\cdots\right]}$$

$$= \lim_{\theta \to 0} \left[\frac{\frac{1}{3!}[n^3-n]+\frac{\theta^2}{5!}[n-n^5]+\cdots}{\frac{1}{2!}[n^2-1]+\frac{\theta^2}{4!}[1-n^4]+\cdots}\right]$$

$$= \frac{1}{6}\left[n^3 - n\right] \times \frac{2}{n^2-1}$$

$$= \frac{n(n^2-1)}{6} \times \frac{2}{n^2-1}$$

$$= \frac{n}{3} \,//$$

$$\therefore \lim_{\theta \to 0}\left[\frac{n\sin\theta - \sin n\theta}{\theta\left[\cos\theta - \cos n\theta\right]}\right] = \frac{n}{3}$$

For what values of a, b, c

$$\lim_{x \to 0}\left[\frac{x(a + b\cos x) - c\sin x}{x^5}\right] = 1$$

Soln :

$$= \lim_{x \to 0}\left[\frac{x(a + b\cos x) - c\sin x}{x^5}\right] = 1$$

$$= \lim_{x \to 0}\left[\frac{x\left(a + b\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right] - c\left[x - \frac{x^3}{3!} + \frac{x^5}{5!}\right]\right.}{x^5}\right] = 1$$

$$= \lim_{x \to 0}\left[x\left(a + \left[b - \frac{bx^2}{2} + \frac{bx^4}{4!} - \left(x + \frac{cx^3}{3!} + \frac{cx^5}{5!}\right]\right] = x^5$$

$$= \lim_{x \to 0}\left[ax + bx - \frac{bx^3}{2!} + \frac{bx^5}{4!} - cx + \frac{cx^3}{3!} - \frac{cx^5}{5!}\right] = x^5$$

$$= x(a+b-c) + x^3\left[\frac{-b}{2!} + \frac{c}{3!}\right] + x^5\left[\frac{b}{4!} - \frac{c}{5!}\right] = x^5$$

Equating the co-efficient of x term

$$a+b-c = 0 \longrightarrow (1)$$

Equating the co-efficient of x^3 term

$$\frac{-b}{2!} + \frac{c}{3!} = 0 \longrightarrow (2)$$

Equating the co-efficient of x^5 term

$$\frac{b}{4!} - \frac{c}{5!} = 1 \longrightarrow (3)$$

from

$$-\frac{b}{2} + \frac{c}{6} = 0 \rightarrow 2$$

$$\frac{-3b+c}{6} = 0$$

$$-3b+c = 0$$

$$c = 3b$$

Solve $c=3b$ in 3 equation

$$\frac{5b-c}{120} = 1$$

$$5b-c = 120$$

$$5b-3b = 120$$

$$2b = 120$$

$$b = \frac{120}{2} = 60$$

$b = 60$

Solve $b = 60$ in (1) equation

$$a + b - c = 0$$

$$a + 60 + 3b = 0$$

$$a + 60 + 3(60) = 0$$

$$a + 60 - 180 = 0$$

$$a - 120 = 0$$

$$a = 120$$

The values of $a = 120$ $b = 60$ $c = 180$.

$a = 120$ $b = 60$ $c = 180$

Prove that the equation $\sin 3\theta = a\sin\theta + b\cos\theta + c$ has six roots and that the sum of the six values of θ which satisfy it is equal to an odd multiple of π?

Soln:

$$\sin 3\theta = a\sin\theta + b\cos\theta + c$$

$$3\sin\theta - 4\sin^3\theta = a\sin\theta + b\cos\theta + c$$

$$3\left[\frac{2t}{1+t^2}\right] - 4\left[\frac{2t}{1+t^2}\right]^3 = a\left[\frac{2t}{1+t^2}\right] + b\left[\frac{1-t^2}{1+t^2}\right] + c$$

$$\frac{6t}{1+t^2} - \frac{32t^3}{(1+t^2)^3} = \frac{2at}{1+t^2} + \frac{b(1-t^2)}{1+t^2} + c$$

$$\frac{6t}{1+t^2} - \frac{32t^3}{(1+t^2)^3} - \frac{2at}{1+t^2} - \frac{b(1-t^2)}{1+t^2} - c = 0$$

$$\frac{6t(1+t^2)^2-32t^3-2at(1+t^2)^2-b(1-t^2)(1+t^2)^2-c(1+t^2)^3}{(1+t^2)^3}=0$$

$$6t[1+t^4+2t^2]-32t^3-2at(1+t^2)^2-b(1-t^2)[1+t^4+2t^2]$$
$$-c[1+3t^4+3t^2+t^6]=0$$

$$6t+6t^5+12t^3-32t^3-2at-2at^5-4at^3-b-bt^4-2bt^2$$
$$+bt^2+bt^6+2bt^4-c-3ct^4-3ct^2-ct^6=0$$

$$t^6(b-c)-2t^5(a-3)+t^4(b-3c)-4t^3(5+a)$$
$$-t^2(b+3c)-2t(a-3)-(b+c)=0$$

This is a 6th degree equation in t.

It has 6 roots $t_1, t_2, t_3, t_4, t_5, t_6$

$$s_1 = \frac{-\text{co efficient of } t^5}{\text{co efficient of } t^6} = \frac{-(-2[a-3])}{b-c}$$

$$s_2 = +\frac{\text{co efficient of } t^4}{\text{co efficient of } t^6} = \frac{b-3c}{b-c}$$

$$s_3 = -\frac{\text{co efficient of } t^3}{\text{co efficient of } t^6} = \frac{-(-4[5+a])}{b-c}$$

$$s_4 = +\frac{\text{co efficient of } t^2}{\text{co efficient of } t^6} = \frac{+\ b+3c}{b-c}$$

$$s_5 = -\frac{\text{co efficient of } t}{\text{co efficient of } t^6} = \frac{-(-2[a-3])}{b-c}$$

$$s_6 = +\frac{\text{co efficient of constant}}{\text{co efficient of } t^6} = \frac{-b+c}{b-c}$$

$$\tan\left[\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+t_6}{2}\right] = \frac{S_1-S_3+S_5}{1-S_2+S_4-S_6}$$

$$= \frac{\frac{2a-6}{b-c} - \frac{4(a+5)}{b-c} + \frac{2(a-3)}{b-c}}{1-\frac{b-3c}{b-c}-\frac{b+3c}{b-c}+\frac{b+c}{b-c}}$$

$$= \frac{2(a-3)-4(a+5)+2(a-3)}{(b-c)-(b-3c)-(b+3c)+(b+c)}$$

$$= \frac{2(a-3)-4(a+5)+2(a-3)}{b-c-b+3c-b-3c+b+c}$$

$$= \frac{2(a-3)-4(a+5)+2(a-3)}{0} = \infty$$

$$\frac{\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6}{2} = \tan^{-1}\infty$$

$$\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6 = 2\left[\frac{\pi}{2}+n\pi\right] \quad n=0,1,2..$$

$$= 2\left[\frac{\pi}{2}+n\pi\right]$$

$$= 2\left[\frac{\pi+2n\pi}{2}\right] = \frac{2\pi}{2}[1+2n]$$

$$= \pi(1+2n)$$

Hence proved// $=$ an odd multiple of π//

Prove that the equation $\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$ has four roots & that the sum of the four values of θ which satisfy it is equal to an odd multiple of π radians.

Soln:

$$\frac{ah}{\left[\frac{1-t^2}{1+t^2}\right]} - \frac{bk}{\left[\frac{2t}{1+t^2}\right]} = a^2 - b^2$$

$$\frac{ah(1+t^2)}{(1-t^2)} - \frac{bk(1+t^2)}{2t} = a^2 - b^2$$

$$\frac{ah(1+t^2)(2t) - bk(1+t^2)(1-t^2)}{2t(1-t^2)} = a^2 - b^2$$

$$ah\,2t + 2aht^3 - bk(1+t^2-t^2-t^4) = a^2-b^2[2t(1-t^2)]$$

$$ah2t + 2aht^3 - bk + bkt^4 = a^2-b^2(2t-2t^3)$$

$$ah2t + 2aht^3 - bk + bkt4 = 2a^2t - 2a^2t^3 - 2b^2t + 2b^2t^3$$

$$ah2t + 2aht^3 - bk + bkt^4 - 2a^2t - 2a^2t^3 - 2b^2t + 2b^2t^3 = 0$$

$$t^4(bk) + t^3(2ah + 2a^2 - 2b^2) + t(2ah - 2a^2 + 2b^2) = 0$$

this is the 4th degree equation

It has 4 roots namely t_1, t_2, t_3, t_4

$$s_1 = -\frac{\text{co efficient of } t^3}{\text{co efficient of } t^4} = \frac{-2(ah+a^2-b^2)}{bk}$$

$$s_2 = +\frac{\text{co efficient of } t^2}{\text{co efficient of } t^4} = 0$$

$$S_3 = -\frac{\text{coefficient of } t}{\text{coefficient of } t^4} = -\frac{2(ah-a^2+b^2)}{bk}$$

$$S_4 = \frac{\text{coefficient of constent}}{\text{coefficient } t^4} = \frac{-bk}{bk} = -1$$

$$\tan\frac{(\theta_1+\theta_2+\theta_3+\theta_4)}{2} = \frac{S_1-S_3}{1-S_2+S_4}$$

$$= \frac{\frac{-2(ah+a^2-b^2)}{bk} + \frac{2(ah-a^2+b^2)}{bk}}{0} = \infty$$

$$\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2} = \tan^{-1}\infty$$

$$\theta_1+\theta_2+\theta_3+\theta_4 = 2\left[\frac{\pi}{2}+n\pi\right] \quad n=0,1,2\ldots$$

$$= 2\left[\frac{\pi}{2}+n\pi\right]$$

$$= 2\left[\frac{\pi+2n\pi}{2}\right]$$

$$= \frac{2\pi}{2}[1+2n]$$

$$= \pi[1+2n] \quad n=0,1,2,\ldots$$

= an odd multiple of π

Hence proved//

Find the equation whose roots are $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{6\pi}{7}$.

Soln:

Put $\frac{2\pi}{7}$ or $\frac{4\pi}{7}$ or $\frac{6\pi}{7} = \theta$

$\text{even } \pi = 7\theta$

$\text{even } \pi = 4\theta + 3\theta$

$4\theta = \text{even } \pi - 3\theta$

$\cos 4\theta = \cos[\text{even } \pi - 3\theta]$

(multiply) $*\cos\theta$

$\cos 4\theta = \cos 3\theta$

$\cos 4\theta = 4\cos^3\theta - 3\cos\theta$

Here $\theta = 2\theta$

$[\cos 2(2\theta)] = 4\cos^3\theta - 3\cos\theta$

$2\cos^2(2\theta) - 1 = 4\cos^3\theta - 3\cos\theta$

$2(2\cos^2\theta - 1)^2 - 1 = 4\cos^3\theta - 3\cos\theta$

$2[4\cos^4\theta + 1 - 4\cos^2\theta] - 1 - 4\cos^3\theta + 3\cos\theta = 0$

$8\cos^4\theta + 2 - 8\cos^2\theta - 1 - 4\cos^3\theta + 3\cos\theta = 0$

$8\cos^4\theta - 4\cos^3\theta - 8\cos^2\theta + 3\cos\theta + 1 = 0$

Take $\cos\theta = x$

$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$

Here $x = 1$

$8(1) - 4(1) - 8(1) + 3(1) + 1 = 0$

$8-4-8+3+1=0$

$\therefore x-1=0$

$$
\begin{array}{r|l}
 & 8x^3+4x^2-4x-1 \\
\hline
x-1 & 8x^4-4x^3-8x^2+3x+1 \\
 & 8x^4-8x^3 \\
\hline
 & \quad 4x^3-8x^2 \\
 & \quad 4x^3-4x^2 \\
\hline
 & \qquad -4x^2+3x \\
 & \qquad -4x^2+4x \\
\hline
 & \qquad\qquad -x+1 \\
 & \qquad\qquad -x+1 \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

$(x-1)[8x^3+4x^2-4x-1]=0$

$8x^3+4x^2-4x-1=0 \longrightarrow (1)$

has roots $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, $\cos\frac{6\pi}{7}$

Put $2x=y$ in (1)

$y^3+y^2-2y-1=0$

has roots $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{6\pi}{7}$ //

Hence proved //

If $\cos(x+iy) = \cos a + i \sin a$

P.T. $\cos 2x + \cos 2hy = 2$

Proof:-

$$\cos a + i \sin a = \cos(x+iy)$$

$$= \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cos hy - i \sin x \sin hy$$

Equating the real & imaginary part.

$$\cos a = \cos x \cosh y \quad — (1), \qquad \sin a = -\sin x \sinh y \quad — (2)$$

squiring,

$$\cos^2 a = \cos^2 x \cosh^2 y, \quad \sin^2 a = \sin^2 x \sinh^2 y$$

on adding

$$\cos^2 a + \sin^2 a = \cos^2 x \cdot \cosh^2 y + \sin^2 x \sinh^2 y$$

$$1 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

$$\left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1-\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right) = 1$$

$(1+\cos 2x)(1+\cosh 2y) + (1-\cos 2x)(\cosh 2y - 1)$

$= 4$

$$\left.\begin{aligned} &1+\cosh 2y + \cancel{\cos 2x} + \cos 2x \cosh 2y \\ &+ \cosh 2y - 1 - \cancel{\cos 2x} \cosh 2y + \cos 2x \end{aligned}\right\} = 4$$

$2\cosh 2y + 2\cos 2x = 4$

$2\left[\cosh 2y + \cos 2x\right] = 4$

$\cosh 2y + \cos 2x = 2$

Hence Proved //

If $\sin(x+iy) = \tan a + i \sec a$

P.T. $\cos 2x \cosh 2y = 3$

Proof :-

$\tan a + i \sec a = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$\tan a + i \sec a = \sin x \cosh y + i \cos x \sinh y$

Equating the real & imaginary parts,

$\tan a = \sin x \cosh y$ — ①, $\sec a = \cos x \sinh y$ — ②

Squaring & subtract ① & ②

$\sec^2 a - \tan^2 a = \cos^2 x \sinh^2 y - \sin^2 x \cosh^2 y$

$$\left(\frac{1+\cos 2x}{2}\right)\left(\frac{\cosh 2y-1}{2}\right)-\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right)=1$$

$$(1+\cos 2x)(\cosh 2y-1)-(1-\cos 2x)(1+\cosh y)=4$$

$$\cosh 2y-1+\cos 2x\cosh 2y-\cos 2x-\left[1+\cosh y-\cos 2x-\cos 2x\cosh y\right]=4$$

$$\left.\begin{aligned}&\cosh 2y-1+\cos 2x\cosh 2y-\cos 2x\\&-1-\cosh y+\cos 2x+\cos 2x\cosh y\end{aligned}\right\}=4$$

$$-2+2\cos 2x\cosh 2y=4$$

$$2\cos 2x\cosh 2y=4+2$$

$$\cos 2x\cosh 2y=6/2$$

$$\cos 2x\cosh 2y=3$$

Hence Proved //

Separate into real & imaginary part of $\tan^{-1}(x+iy)$

Soln:

Let $\tan^{-1}(x+iy)=A+iB$

$$x+iy=\tan(A+iB)$$

$|||^{rly}$ $$x-iy=\tan(A-iB)$$

To find the real part of A

$$\tan 2A = \tan\left[(A+iB)+(A-iB)\right]$$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$\tan 2A = \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2A = \frac{2x}{1-\left[x^2-(iy)^2\right]}$$

$$\tan 2A = \frac{2x}{1-x^2+y^2}$$

$$2A = \tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

$$A = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

Find the imaginary part of B

consider $2B = (A+iB)-(A-iB)$

$$\tan 2B = \tan\left[(A+iB)-(A+iB)\right]$$

$$= \frac{\tan(A+iB)-\tan(A+iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$\tan 2B = \frac{x+iy-x+iy}{1+(x+iy)(x-iy)}$$

$$\tan 2iB = \frac{2iy}{1+[x^2-(iy)^2]}$$

$$i\tanh 2B = \frac{2iy}{1+x^2+y^2}$$

$$2B = \tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

$$B = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

Expand $\cosh^5\theta$ in terms of multiple angles of hyperbolic cosine :

Soln:

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

$$(\cosh\theta)^5 = \left(\frac{e^{\theta}+e^{-\theta}}{2}\right)^5 = \frac{1}{2^5}\left(e^{\theta}+e^{-\theta}\right)^5$$

$$a = e^{\theta}, \quad b = e^{-\theta}, \quad n=5$$

$$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + nC_3 a^{n-3} b^3 + nC_4 a^{n-4} b^4 + nC_5 a^{n-5} b^5$$

$$(e^{\theta}+e^{-\theta})^5 = e^{5\theta} + 5C_1 e^{4\theta} e^{-\theta} + 5C_2 e^{3\theta} e^{-2\theta} + 5C_3 e^{2\theta} e^{-3\theta} + 5C_4 e^{\theta} e^{-4\theta} + 5C_5 e^{-5\theta}$$

$$= e^{5\theta} + 5e^{4\theta}e^{-\theta} + 10e^{3\theta}e^{-2\theta} + 10e^{2\theta}e^{-3\theta} + 5e^{2\theta}e^{-4\theta} + e^{-5\theta}$$

$$= \frac{1}{2^5}\left(e^{5\theta} + 5e^{4\theta}e^{-\theta} + 10e^{3\theta}e^{-2\theta} + 10e^{2\theta}e^{-3\theta} + 5e^{2\theta}e^{-4\theta} + e^{-5\theta}\right)$$

$$= \frac{1}{2^5}\left(e^{5\theta} + 5e^{3\theta} + 10e^{\theta} + 10e^{-\theta} + 5e^{-3\theta} + e^{-5\theta}\right)$$

$$= \frac{1}{2^5}\left[e^{5\theta} + 5e^{3\theta} + 10e^{\theta} + 10\left(\frac{1}{e^{\theta}}\right) + 5\left(\frac{1}{e^{3\theta}}\right) + \left(\frac{1}{e^{5\theta}}\right)\right]$$

$$= \frac{1}{2^5}\left[\left(e^{5\theta} + \frac{1}{e^{5\theta}}\right) + 5\left(e^{3\theta} + \frac{1}{e^{3\theta}}\right) + 10\left(e^{\theta} + \frac{1}{e^{\theta}}\right)\right]$$

$$= \frac{1}{2^5}\left[2\cosh 5\theta + 5(2\cosh 3\theta) + 10(2\cosh\theta)\right]$$

$$= \frac{2}{2^5}\left[\cosh 5\theta + 5\cosh 3\theta + 10\cosh\theta\right]$$

$$= \frac{1}{2^4}\left[\cosh 5\theta + 5\cosh 3\theta + 10\cosh\theta\right]$$

Expand $\sinh^6\theta$ in terms of multiple angle of hyperbolic cosine:

Soln:

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$(\sinh\theta)^6 = \left(\frac{e^{\theta} - e^{-\theta}}{2}\right)^6$$

$$\sinh^6\theta = \left(\frac{e^{\theta}-e^{-\theta}}{2}\right)^6 \Rightarrow \frac{1}{2^6}\left(e^{\theta}-e^{-\theta}\right)^6$$

$$(a-b)^n = a^n - nC_1 a^{n-1}b + nC_2 a^{n-2}b^2 - nC_3 a^{n-3}b^3 + nC_4 a^{n-4}b^4 - nC_5 a^{n-5}b^5 + nC_6 a^{n-6}b^6$$

$$a = e^{\theta} \quad b = e^{-\theta} \quad n = 6$$

$$(e^{\theta}-e^{-\theta})^6 = e^{6\theta} - 6C_1 e^{5\theta}e^{-\theta} + 6C_2 e^{4\theta}e^{-2\theta} - 6C_3 e^{3\theta}e^{-3\theta} + 6C_4 e^{2\theta}e^{-4\theta} - 6C_5 e^{\theta}e^{-5\theta} + 6C_6 e^{0}e^{-6\theta}$$

$$= e^{6\theta} - 6e^{5\theta}e^{-\theta} + 15e^{4\theta}e^{-2\theta} - 15e^{3\theta}e^{-3\theta} + 15e^{2\theta}e^{-4\theta} - 6e^{\theta}e^{-5\theta} + e^{-6\theta}$$

$$= \frac{1}{2^6}\left(e^{6\theta} - 6e^{5\theta}e^{-\theta} + 15e^{4\theta}e^{-2\theta} - 20e^{3\theta}e^{-3\theta} + 15e^{2\theta}e^{-4\theta} - 6e^{\theta}e^{-5\theta} + e^{-6\theta}\right)$$

$$= \frac{1}{2^6}\left(e^{6\theta} - 6e^{4\theta} + 15e^{2\theta} - 20e^{0} + 15e^{-2\theta} - 6e^{-4\theta} + e^{-6\theta}\right)$$

$$= \frac{1}{2^6}\left[\cancel{2\cos h}\left[\left(e^{6\theta}+\frac{1}{e^{6\theta}}\right) - 6\left(e^{4\theta}+\frac{1}{e^{4\theta}}\right) + 15\left(e^{2\theta}+\frac{1}{e^{2\theta}}\right) - 20\right]\right.$$

$$= \frac{1}{2^6}\left[2\cosh 6\theta - 6(2\cosh 4\theta) + 15(2\cosh 2\theta) - 20\right]$$

$$= \frac{2}{2^6}\left[\cosh 6\theta - 6\cosh 4\theta + 15\cosh 2\theta - 10\right]$$

$$= \frac{1}{2^5}\left[\cosh 6\theta - 6\cosh 4\theta + 15\cosh 2\theta - 10\right]$$

P.T. $\sinh^{-1} x = \log_e x + \sqrt{x^2+1}$

proof :-

Let $y = \sinh^{-1} x$

$$x = \sinh y$$

$$= \frac{e^y - e^{-y}}{2}$$

$e^y \times e^{-y} - e^{-y}$

$e^{-y}(e^y - 1)$

$$= \frac{e^{-y}\left(\frac{e^y}{e^{-y}} - 1\right)}{2}$$

$$= \frac{e^{-y}\left[e^{2y} - 1\right]}{2}$$

$$x = \frac{e^{2y} - 1}{2e^y}$$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$2xe^y = e^{2y} - 1$$

$a = 1, \quad b = -2x, \quad c = -1$

$$e^{2y} - 2xe^y - 1 = 0$$

This is a quadratic equation in e^y.

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \qquad e^y = \frac{2 \times 2\sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$= \frac{\cancel{2}(x \pm \sqrt{x^2+1})}{\cancel{2}}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Take log, on both sides.

$$\log e^y = \log(x \pm \sqrt{x^2+1})$$

$$y = \log(x \pm \sqrt{x^2+1})$$

$$\sinh^{-1} x = \log(x \pm \sqrt{x^2+1}) //$$

P.T. $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

Soln:

Let $y = \tanh^{-1} x$

$$x = \tanh y$$

$$= \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x = \frac{e^y(e^y - e^{-y})}{e^y(e^y + e^{-y})}$$

$$= \frac{e^y e^y - e^y e^{-y}}{e^y e^y + e^y e^{-y}}$$ (x^{ly} e^y both in the num & deno)

$$= \frac{e^{2y} - 1}{e^{2y} + 1} \quad (\because e^1 = 1)$$

$$x(e^{2y}+1) = e^{2y} - 1$$

$$x e^{2y} + x = e^{2y} - 1$$

$$\quad 2y \qquad 2y$$

$$x+1 = e^{2y}(1-x)$$

$$e^{2y} = \frac{x+1}{1-x}$$

Taking log on both sides,

$$2y = \log \frac{x+1}{1-x}$$

$$y = \frac{1}{2} \log \frac{1+x}{1-x}$$

Hence proved //

If $(u+iv) = \cosh(x+iy)$

P.T. $\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$ & $\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 x} = 1$

Soln:-

$$u+iv = \cosh(x+iy)$$

$$= \cos i(x+iy)$$

$$= \cos(xi-y)$$

$$= \cos xi \cos y + \sin xi \sin y$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$$u = \cosh x \cos y \;\&\; v = \sinh x \sin y$$

(i) $$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1$$

$$= \frac{(\cosh x \cos y)^2}{\cosh^2 x} + \frac{(\sinh x \sin y)^2}{\sinh^2 x}$$

$$= \frac{\cancel{\cosh^2 x} \cos^2 y}{\cancel{\cosh^2 x}} + \frac{\cancel{\sinh^2 x} \sin^2 y}{\cancel{\sinh^2 x}}$$

$$= \cos^2 y + \sin^2 y = 1$$

Hence proved //

(ii) $$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 x} = 1$$

$$= \frac{(\cos x \cos y)^2}{\cosh^2 y} - \frac{(\sin x \sin y)^2}{\sinh^2 y}$$

$$= \frac{\cos^2 x \cancel{\cosh^2 y}}{\cancel{\cosh^2 y}} - \frac{\sin^2 x \cancel{\sinh^2 y}}{\cancel{\sinh^2 y}}$$

$$= \cos^2 x - \sin^2 x = 1$$

Hence proved //

①

I. If $x+iy = \cos(u+iv)$, where x, y, u, v are real

P.T $(1+x)+y^2 = (\cosh v+\cos u)^2$ &

$(1-x)^2+y^2 = (\cosh v - \cos u)^2$

Soln

$$x+iy = \cos(u+iv)$$

$$= \cos u \cos iv - \sin u \cdot \sin iv$$

$$= \cos u \cosh v - i \sin u \sinh v.$$

$x = \cos u \cosh v$

$y = -\sin u \sinh v$

L.H.S

$$\text{i)} (1+x)^2+y^2 = [1+\cos u+\cosh v]^2 + [-\sin^2 u \sinh^2 v]$$

$$= 1+\cos^2 u \cosh^2 v + 2\cos u \cosh v + \sin^2 u \sinh^2 v$$

$$= 1+\cos^2 u \cosh^2 v + 2\cos u \cosh v + (1-\cos^2 u)(\cosh^2 u - 1)$$

$$= \cancel{1}+\cancel{\cos^2 u} \cancel{\cosh^2 v} + 2\cos u \cosh v + \cosh^2 u \cancel{-1} - \cancel{\cos^2 u} \cancel{\cosh^2 u} + \cos^2 u.$$

$$= (\cosh v + \cos u)^2$$ Hence proved.

(ii) $(1-x)^2+y^2 = 1-(\cosh v - \cos u)^2 + \sin^2 u \sinh^2 v$

$$= 1-(\cosh^2 + \cos^2 u - 2\cosh v \cos u) + (1-\cos^2 u)(\cosh^2 v - 1)$$

$$= \cancel{1} - \cosh^2 v + \cos^2 u - 2\cosh v \cos u + \cosh^2 v - \cancel{1} - \cos^2 u$$

$$\cancel{\cosh^2 v} + \cos^2 u$$

$$= - \cosh^2 v$$

$$= (\cosh v - \cos u)^2$$ Hence proved.

If $\tan(\theta + i\phi) = \sin(x + iy)$.

P.T $\cosh y \sinh 2\phi = \cot x \sin 2\phi$.

Soln.

Given $\tan(\theta + i\phi) = \sin(x+iy)$

$$\frac{\sin(\theta+i\phi)}{\cos(\theta+i\phi)} = \sin x \cos iy + \cos x \sin iy$$

Taking conjugate in the L.H.S.

$$\frac{2\sin(\theta+i\phi)\cos(\theta-i\phi)}{2\cos(\theta+i\phi)\cos(\theta-i\phi)} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta + \sin 2i\phi}{\cos 2\theta + \cos 2i\phi} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta + i\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y + i\cos x \sinh y$$

$$\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\theta} + \frac{i\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y + i\cos x \sinh y$$

Equating the real & imaginary part.

$$\frac{\sin 2\theta}{\cos 2\theta + \cosh 2\phi} = \sin x \cosh y \quad \text{—} (1)$$

$$\frac{\sinh 2\phi}{\cos 2\theta + \cosh 2\phi} = \cos x \sinh y \quad \text{—} (2)$$

Dividing (1) & (2)

$$\tan x \coth y = \frac{\sin 2\theta}{\sinh 2\phi}.$$

$$\coth y \sinh 2\phi = \frac{1}{\tan x} \sin 2\theta.$$

$\coth y \sinh 2\phi = \cot x \sin 2\theta$. Hence proved.

If $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

PT $r^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$ &

$\tan\alpha = \tanh\phi \cot\theta$.

soln Given $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$.

$\sin\theta \cos i\phi + \cos\theta \sin i\phi = r(\cos\alpha + i\sin\alpha)$.

$\sin\theta \cosh\phi + \cos\theta \sinh\phi = r\cos\alpha + i r\sin\alpha$.

Real part $= r\cos\alpha = \sin\theta \cosh\phi$ —(1)

Imaginary part $= r\sin\alpha = \cos\theta \sinh\phi$ —(2)

squaring adding (1) & (2).

$r^2\cos^2\alpha + r^2\sin^2\alpha = \sin^2\theta \cosh^2\phi + \cos^2\theta \sinh^2\phi$.

$$r^2 = \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cosh 2\phi}{2}\right) + \left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cosh 2\phi - 1}{2}\right)$$

$$r^2 = \frac{1}{4}\left[(1-\cos 2\theta)(1+\cosh 2\phi) + (1+\cos 2\theta)(\cosh 2\phi - 1)\right]$$

$$= \frac{1}{4}\left[1+\cosh 2\phi - \cos 2\theta - \cos 2\theta \cosh 2\phi + \cosh 2\phi - 1 + \cos 2\theta \cosh 2\phi - \cos 2\theta\right]$$

$$= \frac{1}{4}\left[2\cosh 2\phi - 2\cos 2\theta\right]$$

$$= \frac{2}{4}\left[\cosh 2\phi - \cos 2\theta\right]$$

$$r^2 = \frac{1}{2}\left[\cosh 2\phi - \cos 2\theta\right]$$

If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$

PT $\quad y = \frac{1}{2}\log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$

Soln

$\cos x \cos iy - \sin x \sin iy = r\cos\alpha + r i\sin\alpha.$

$\cos x \cosh y - \sin x \sinh y = r\cos\alpha + r i\sin\alpha.$

Real part $= \cos x \cosh y = r\cos\alpha$

Imaginary part $= i\sin x \sinh y = i r\sin\alpha.$

Dividing ② by ①

$$\frac{\cancel{r}\sin\alpha}{\cancel{r}\cos\alpha} = -\frac{\sin x \sinh y}{\cos x \cosh y}$$

$$\frac{\cosh y}{\sinh y} = \frac{-\sin x \cos\alpha}{\cos x \sin\alpha}$$

$$\cot hy = \frac{-\sin x \cos\alpha}{\cos x \sin\alpha}.$$

$$\frac{e^{x}+e^{-y}}{e^{x}-e^{-y}} = \frac{-\sin x \cos\alpha}{\cos x \sin\alpha}$$

By componando & dividedo rule

$$\frac{(e^{y}+e^{-y})+(e^{y}-e^{-y})}{(e^{y}+e^{-y})-(e^{y}-e^{-y})} = \frac{-\sin x \cos\alpha + \cos x \sin\alpha}{-\sin x \cos\alpha - \cos x \sin\alpha}.$$

$$\frac{\cancel{2}e^{y}}{\cancel{2}e^{-y}} = \frac{-(\sin x \cos\alpha - \cos x \sin\alpha)}{-(\sin x \cos\alpha + \cos x \sin\alpha)}$$

$$\frac{e^{y}}{e^{-y}} = \frac{\sin(x-\alpha)}{\sin(x+\alpha)}$$

$$e^{y}.e^{-y} = \frac{\sin(x-\alpha)}{\sin(x+\alpha)}$$

$$e^{2y} = \frac{\sin(x-\alpha)}{\sin(x+\alpha)}$$

Taking log, on both sides.

$$2y = \log\left[\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right]$$

$$y = \frac{1}{2}\left[\frac{\sin(x-\alpha}{\sin(x+\alpha}\right]$$ Hence proved.

If $\sin(\theta+i\phi) = \cos\alpha + i\sin\alpha$.

PT $\cos^2\theta = \pm\sin\alpha$.

soln Given $\sin(\theta+i\phi) = \cos\alpha + i\sin\alpha$.

$\sin\theta \cos i\phi + \cos\theta \sin i\phi = \cos\alpha + i\sin\alpha$.

$\sin\theta \cosh\phi + \cos\theta \sinh\phi = \cos\alpha + i\sin\alpha$.

Equating real & Imaginary part.

$\sin\theta \cosh\phi = \cos\alpha$.

$\cos\theta \sinh\phi = \sin\alpha$.

$\cosh\phi = \dfrac{\cos\alpha}{\sin\theta}$

$\sinh\phi = \dfrac{\sin\alpha}{\cos\theta}$.

W.K.T

$\cosh^2\phi - \sinh^2\phi = 1$

$\left(\dfrac{\cos\alpha}{\sin\theta}\right)^2 - \left(\dfrac{\sin\alpha}{\cos\theta}\right)^2 = 1$

$\dfrac{\cos^2\alpha}{\sin^2\theta} - \dfrac{\sin^2\alpha}{\cos^2\theta} = 1$

$\dfrac{\cos^2\alpha \cos^2\theta - \sin^2\alpha \sin^2\theta}{\sin^2\theta \cos^2\theta} = 1$

$\cos^2\alpha \cos^2\theta - \sin^2\alpha \sin\theta = \sin^2\theta \cos^2\theta$.

$(1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta) = (1-\cos^2\theta)\cos^2\theta)$.

$\cos^2\theta - \sin^2\alpha\cos^2\theta - \sin^2\alpha + \sin^2\alpha\cos^2\theta = \cos^2\theta - \cos^4\theta$.

$\cos^2\theta - \sin^2\alpha - \cos^2\theta = -\cos^4\theta$.

$-\sin^2\alpha = -\cos^4\theta$.

Taking squring root

$\pm\sin\alpha = \cos^2\theta$ //.

If $\cos(\alpha+i\beta) = \cos\theta + i\sin\theta$.

PT $\sin^2\alpha = \pm \sin\theta$.

Soln Given $\cos(\alpha+i\beta) = \cos\theta + i\sin\theta$.

$\cos\alpha \sin i\beta + \sin\alpha \cos i\beta = \cos\theta + i\sin\theta$.

$\cos\alpha \sinh\beta + \sin\alpha \cosh\beta = \cos\theta + i\sin\theta$.

real, Imaginary part.

$$\cos\alpha \sinh\beta = \cos\theta \quad \text{——} (1)$$

$$\sin\alpha \cosh\beta = \sin\theta \quad \text{——} (2)$$

$$\sinh\beta = \frac{\cos\theta}{\cos\alpha} \quad ;$$

$$\cosh\beta = \frac{\sin\theta}{\sin\alpha} \quad ;$$

W.K.T. $\cosh^2\beta - \sinh^2\beta = 1$

$$\left(\frac{\cos\theta}{\cos\alpha}\right)^2 - \left(\frac{\sin\theta}{\sin\alpha}\right)^2 = 1$$

$$\left(\frac{\cos^2\theta}{\cos^2\alpha}\right) - \left(\frac{\sin^2\theta}{\sin^2\alpha}\right) = 1.$$

$$\frac{\cos^2\theta \sin^2\alpha - \sin^2\theta \cos^2\alpha}{\cos^2\alpha \sin^2\alpha} = 1$$

$\cos^2\theta \sin^2\alpha - \sin^2\theta \cos^2\alpha = \cos^2\alpha \sin^2\alpha$.

$(1-\sin^2\theta)\sin^2\alpha - \sin^2\theta(1-\sin^2\alpha) = \cos^2\alpha \sin^2\alpha$.

$\sin^2\alpha - \cancel{\sin^2\theta \sin^2\alpha} - \sin^2\theta + \cancel{\sin^2\theta \sin^2\alpha} = \cos^2\alpha \sin^2\alpha$.

$$\sin^2\alpha - \sin^2\theta = (1-\sin^2\alpha)\sin^2\alpha.$$

$$\sin^2\alpha - \sin^2\theta = \sin^2\alpha - \sin^4\alpha.$$

$$\cancel{\sin^2\alpha} - \sin^2\theta - \cancel{\sin^2\alpha} = -\sin^4\alpha.$$

$$-\sin^2\theta = -\sin^4\alpha.$$

taking squaring root.

$$\pm \sin\theta = \sin^2\alpha$$

If $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$.

soln.

$$2\theta = n\pi + \frac{\pi}{2} + \alpha.$$

$$e^{2\theta} = \pm \cot\frac{\alpha}{2}.$$

soln

Given $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha.$

$$\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha.$$

$$2\theta = (\theta + i\phi) + (\theta - i\phi)$$

$$\tan 2\theta = \tan(\theta + i\phi) + (\theta - i\phi)$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$= \frac{\tan\alpha + i\sec\alpha + \tan\alpha - i\sec\alpha.}{1 - [(\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)]}$$

$$= \frac{2\tan\alpha.}{1 - [\tan^2\alpha - i\sec^2\alpha]}$$

$$\tan 2\theta = \frac{2\tan\alpha.}{1 - \tan^2\alpha + \sec^2\alpha.}$$

$$= \frac{2\tan\alpha}{1-\tan^2\alpha-(1+\tan^2\alpha)}$$

$$= \frac{2\tan\alpha}{-2\tan^2\alpha}$$

$$= \frac{1}{\tan\alpha}$$

$$\tan 2\theta = -\cot\alpha$$

$$\tan 2\theta = \tan(\pi/2+\alpha)$$

$$2\theta = \pi/2+\alpha$$

i.e. $2\theta = n\pi + \pi/2 + \alpha$ //.

now! $\quad 2i\phi = (\theta+i\phi)-(\theta-i\phi)$

$$\tan 2i\phi = \tan[(\theta+i\phi)-(\theta-i\phi)]$$

$$= \frac{\tan(\theta+i\phi)-\tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\tan\alpha+i\sec\alpha-(\tan\alpha+i\sec\alpha)}{1+(\tan\alpha+i\sec\alpha)(\tan\alpha-i\sec\alpha)}$$

$$= \frac{2i\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$$

$$= \frac{2i\sec\alpha}{\sec^2\alpha+\sec^2\alpha} \quad [\because 1+\tan^2\alpha=\sec^2\alpha]$$

$$= \frac{\cancel{2}i\sec\alpha}{\cancel{2}\sec^2\alpha}$$

$$= i\frac{1}{\sec^2\alpha}$$

$$\tan 2i\phi = i\cos\alpha.$$

$$\cancel{i}\tanh 2\phi = \cancel{i}\cos\alpha.$$

$$\tanh 2\phi = \cos\alpha.$$

$$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \cos\alpha.$$

By componendo & dividendo rule.

$$\frac{(e^{2\phi}+e^{-2\phi})(e^{+2\phi}+e^{-2\phi})}{(e^{2\phi}-e^{-2\phi})(e^{2\phi}+e^{-2\phi})} = \frac{\cos\alpha+1}{\cos\alpha-1}$$

$$\frac{2e^{2\phi}}{-2e^{-2\phi}} = \frac{(1+\cos\alpha)}{-(1-\cos\alpha)}$$

$$\frac{e^{2\phi}}{e^{-2\phi}} = 2\cos^2\alpha/2$$

$$e^{2\phi}\cdot e^{-2\phi} = \cot^2\alpha/2$$

$$(e^{2\phi})^2 = \cot^2\alpha/2$$

$$e^{2\phi} = \pm\cot\alpha/2,,$$

If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$

S.T $\theta = n\pi/2 + \pi/4$ &.

$\phi = \frac{1}{2}\log\tan\left(\pi/4 + \alpha/2\right)$,,

Soln

Given $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$.

$\tan(\theta-i\phi) = \cos\alpha - i\sin\alpha$.

$$2i\phi = [(\theta+i\phi) - (\theta-i\phi)]$$

$$= \frac{\tan(\theta+i\phi) - \tan(\theta-i\phi)}{1+\tan(\theta+i\phi)\tan(\theta-i\phi)}.$$

$$= \frac{\cos\alpha + i\sin\alpha - \cos\alpha - i\sin\alpha}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha \pm i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1+\cos^2\alpha - i\sin^2\alpha}$$

$$= \frac{2i\sin\alpha}{\sin^2\alpha - i\sin^2\alpha} \qquad = \frac{2i\sin\alpha}{\sin^2\alpha + \sin^2\alpha}$$

$$= \frac{\cancel{2}i\sin\alpha}{\cancel{2}\sin^2\alpha}$$

$$= i\,\frac{1}{\sin^2\alpha}$$

$\tan 2\phi = i\cos\alpha$

$\tanh 2\phi = /i\cos\alpha$

$\tanh 2\phi = \cos\alpha$

$$\frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By Compoundo & dividendo rule.

$$\frac{e^{2\phi} - e^{-2\phi} + e^{2\phi} + e^{-2\phi}}{e^{2\phi} - e^{-2\phi} - (e^{2\phi} + e^{-2\phi})} = \frac{\sin\alpha + 1}{\sin\alpha - 1}$$

$$\frac{2e^{2\phi}}{-2e^{-2\phi}} = \frac{1+\sin\alpha}{-(1-\sin\alpha)}$$

$$\frac{e^{2\phi}}{e^{-2\phi}} = 2\sin^2\alpha/2$$

$$(e^{2\phi})^2 = \frac{\sin\alpha + 1}{1 - \sin\alpha}.$$

$$= \frac{2\sin\alpha/2 \cos\alpha/2 + \cos^2\alpha/2 + \sin^2\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2 \cos\alpha/2}$$

$$= \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2}$$

$$e^{2\phi} = \frac{\cos\alpha/2 + \sin\alpha/2}{\cos\alpha/2 - \sin\alpha/2}$$

$$= \frac{\cos\alpha/2 \left(1 + \frac{\sin\alpha/2}{\cos\alpha/2}\right)}{\cos\alpha/2 \left(1 - \frac{\sin\alpha/2}{\cos\alpha/2}\right)}$$

$$= \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}$$

$$= \frac{\tan\pi/4 + \tan\alpha/2}{1 - \tan\pi/4 \tan\alpha/2}$$

$$e^{2\phi} = \tan(\pi/4 + \alpha/2)$$

taking log side.

$$2\phi = \log \tan(\pi/4 + \alpha/2)$$

$$\phi = \tfrac{1}{2} \log \tan(\pi/4 + \alpha/2)$$ Hence proved.